University of Illinois at Chicago
Department of Physics

Thermodynamics and Statistical Physics Qualifying Exam

January 5, 2007
9:00am-12:00pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted towards the exam's total score.

Mathematical Formulae

Notation:

$$\beta = \frac{1}{k_B T}$$

$$\mathrm{erfc}\,(\mathrm{z}) = \frac{2}{\sqrt{\pi}} \int_{\mathrm{z}}^{\infty} \mathrm{dx}\,\mathrm{exp}\left(-\mathrm{x}^2\right)$$

erfc is known as the complimentary error function

Integrals:

$$\int dx \ln x = x \ln x - x$$

$$\int \frac{dx}{x} = \ln x$$

$$\int_b^{\infty} dx\, \mathrm{exp}\left(-ax^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{a}}\mathrm{erfc}\left(\sqrt{\mathrm{a}}\mathrm{b}\right)$$

$$\int_0^a dx \mathrm{erfc}\,(\mathrm{x}) = \frac{1 - \mathrm{exp}\left(-\mathrm{a}^2\right)}{\sqrt{\pi}} + \mathrm{aerfc(a)}$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad \text{for} \quad x < 1$$

$$\mathrm{exp}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\mathrm{erfc(x)} = \mathrm{exp}\left(-\mathrm{x}^2\right)\left[\frac{1}{\sqrt{\pi}\mathrm{x}} + \ \ldots\right] \qquad \text{for x} \to \infty$$

$$\sinh(x) = x + \ ... \qquad \text{for } x \to 0$$

$$\cosh(x) = 1 + \ ... \qquad \text{for } x \to 0$$

1. Consider a system consisting of N non-interacting particles each with spin $S = 1$ in an external magnetic field, H. For $H = 0$, all of a single particle's spin projections, S_z, are degenerate with energy $E = 0$.

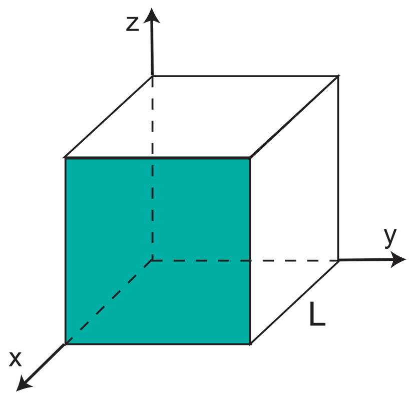
   a) Plot the energies of ALL spin projections, S_z, of a single particle as a function of H.

   b) Calculate the partition function of the system as a function of temperature T and H.

   c) Compute the average energy, $\langle E \rangle$ of the system. What is the form of $\langle E \rangle$ in the limit $T \to 0$? Briefly explain what this result implies for the spin states of the spins (1-2 sentences maximum). What is the form of $\langle E \rangle$ in the limit $\beta g \mu_B H \ll 1$, where $\beta = 1/k_B T$, g is the gyromagnetic ratio, and μ_B is the Bohr magneton?

   d) Compute the specific heat of the system in the high temperature limit for constant H.

2. Consider a three-dimensional box with sides of length L, as shown below



   It contains an ideal gas of non-interacting spin-less particles each with kinetic energy

$$\varepsilon = \frac{m}{2} \vec{v}^2$$

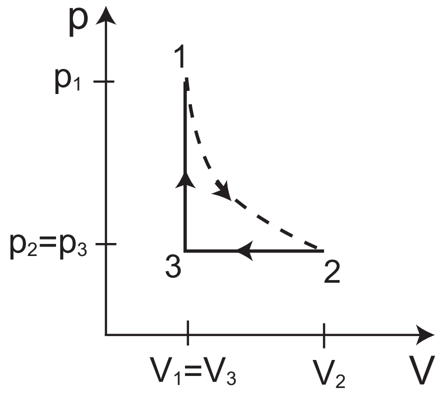
   The temperature of the gas is T, and the particles are uniformly distributed throughout the box.

   a) What is the normalized velocity distribution of the gas?

   b) We now open the front side of the box (the shaded side facing the $+x$-direction, as shown in the figure) for a given time Δt. Using the result from (a), compute the number of particles that escape from the box in time Δt. To this end, consider these two steps: (i) Divide the box into slices of width dx and compute first the number of particles in a given slice at a distance x from the opening that have escape through the opening in time Δt. (ii) In order to find the total number of escaped particles, integrate the result you obtained in (i).

   c) How does the total number of escaped particles depend on Δt in the limit $\Delta t \to 0$?

3. Suppose one mole of an ideal gas is subjected to the cyclic process shown below (with temperature T_1,T_2 and T_3 in states $1, 2$ and 3, respectively)



$1 \Rightarrow 2$ is a free adiabatic expansion, i.e. an expansion against zero applied pressure (like expanding into a vacuum).

$2 \Rightarrow 3$ is a constant pressure compression step

$3 \Rightarrow 1$ is a constant volume heating step

Step $1 \Rightarrow 2$ is irreversible, but steps $2 \Rightarrow 3$ and $3 \Rightarrow 1$ are reversible

a) What is the change in internal energy, ΔU, for the entire cyclic process $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

b) Use the First Law of Thermodynamics to calculate ΔU for the process $1 \Rightarrow 2$.

c) Use the First Law of Thermodynamics to calculate ΔU for the process $2 \Rightarrow 3$.

d) Use the First Law of Thermodynamics to calculate ΔU for the process, $3 \Rightarrow 1$.

e) Using your answers to parts (a) – (d), show that the following result is obtained for 1 mole of an ideal gas:

$$C_p - C_V = R$$

where $C_{V.}$ is the specific heat for constant volume, $C_{p.}$ is the specific heat for constant pressure, and R is the ideal gas constant.

4. Consider a system consisting of M non-interacting molecules at temperature T. Each of these molecules exhibits vibrations with energies

$$E_n = \hbar\omega_0 \left(n + \frac{1}{2}\right) \qquad \text{where} \quad n = 0, 1, 2, 3, ...$$

a) Show that the single particle partition function is given by

$$Z_1 = \frac{1}{2 \sinh \left[\frac{\beta\hbar\omega_0}{2}\right]}$$

and compute the partition function, Z_M, for the M molecule system.

b) Compute the free energy F.and the average energy $\langle E \rangle$ the entire system.What is the form of $\langle E \rangle$ at high temperatures?

c) Compute the entropy S and the specific heat C_V of the system.

5. Consider a monoatomic ideal gas.

   a) What is the internal energy and the equation of state of an ideal gas?

   b) Compute the entropy of an ideal gas as a function of T and V for constant particle number N starting from

$$dU = TdS - pdV$$

   c) Compute the chemical potential of the ideal gas as a function of p and T starting from the Gibbs-Duhem relation

$$SdT - Vdp + Nd\mu = 0$$