Qualifying Exam on Quantum Mechanics, UIC Physics, Jan 2007

Useful Integrals: $\int_0^{\pi/2} \sin^2\theta \; d\theta = \pi/4$, $\int_0^{\pi/2} \theta \sin(2\theta)\cos(3\theta) d\theta = -\frac{12}{25}$.

$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\pi/\alpha}$, $\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2} dx = (-d/d\alpha)^n \sqrt{\pi/\alpha}$.

Problem 1. An electron is confined in a one-dimensional *infinite* potential well in the x range $[-a, +a]$,

$$V(x) = \begin{cases} 0\ , & \text{for } |x| \le a\ , \\ \infty & \text{otherwise.} \end{cases}$$

The initial quantum wave function at $t = 0$ is

$$\psi(x,0) = C\left(3\sin\frac{\pi x}{a} + 4\cos\frac{3\pi x}{2a}\right)$$

1. Determine the normalization constant C.
2. Write down energy eigenvalues and wavefunctions for the first three lowest states. Determine the probability at each energy level for the given $\psi(x,0)$.
3. Determine the mean energy $\langle E\rangle$ for the given $\psi(x,0)$.
4. Find its mean position $\langle x\rangle_0$ at $t = 0$.
5. Determine how the wave function $\psi(x,t)$ changes as time elapses.
6. Calculate its mean position $\langle x\rangle_t$ at any given time t. Determine how long t it takes for the mean position flips sign, i.e. $\langle x\rangle_t$ become $-\langle x\rangle_0$.

Problem 2. A particle of mass m moves in a 1-dimensional potential

$$V(x) = \frac{\hbar^2\kappa_0^2}{m}\left(1 - e^{-\lambda^2 x^2}\right)\ .$$

The parameter κ_0^2 is proportional to the depth of the well. The other parameter λ is inversely proportional to the width of the well.

(a) For a trial wave function

$$\psi(x;\beta) = N e^{-\frac{1}{2}\beta^2 x^2}\ ,$$

show that the energy expectation value has the form

$$E(\beta) = \frac{\hbar^2}{m}\left(A\beta^2 + B\kappa_0^2\frac{\beta}{\sqrt{\beta^2+\lambda^2}} + C\kappa_0^2\right)\ ,$$

where A, B, and C are pure numbers. Determine their values.

(b) In the limit of a deep and wide potential well, $\kappa_0 \to \infty$ and $\lambda \to 0$, but their product is fixed and finite, $\kappa_0 \lambda \equiv K^2$. Then, the energy expectation value has the following expansion,

$$E(\beta) = \frac{\hbar^2}{m}\left(A\beta^2 - \frac{BK^4}{\beta^2} + (B+C)\kappa_0^2\right)$$

for $\beta \gg \lambda$. Find the optimal choice of β for the ground state in terms of K? Estimate the ground state energy.

(c) If we approximate $V(x)$ near the origin by a simple harmonic oscillator potential, what will be the corresponding ground energy in terms of K and m. How is this result in comparison with that of (b)?

Problem 3. Three distinct spin-$\frac{1}{2}$ objects $\mathbf{s}_1$, $\mathbf{s}_3$, $\mathbf{s}_3$, couple each other with the interaction defined by the unperturbed Hamiltonian,

$$\mathcal{H}_0 = A(\mathbf{s}_1 \cdot \mathbf{s}_2 + \mathbf{s}_2 \cdot \mathbf{s}_3 + \mathbf{s}_3 \cdot \mathbf{s}_1) \ .$$

(a) Find energy eigenvalues. Determine the degeneracy of each energy state. Find the sum of energy eigenvalues of eight eigenstates.

Now we modify $\mathcal{H}_0$ such that the new Hamiltonian becomes

$$\mathcal{H} = A(2\mathbf{s}_1 \cdot \mathbf{s}_2 + \mathbf{s}_2 \cdot \mathbf{s}_3 + \mathbf{s}_3 \cdot \mathbf{s}_1) \ ,$$

(b) List a good set of quantum numbers to label the energy eigenstates.

(c) Determine the energy eigenvalues and eigenstates of $\mathcal{H}$. Find the sum of energy eigenvalues of eight eigenstates.

Problem 4. We study a weak ($g \ll 1$) repulsive potential defined by

$$V(r) = \begin{cases} \frac{\hbar^2 g}{2mr^2} & \text{if } r \le R \\ 0 & \text{otherwise} \ . \end{cases}$$

Here R is the range of the potential interaction.

(a) Use the Born approximation $f_{\mathbf{k}}(\hat{\mathbf{k}'}) = -\frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'}V(\mathbf{r}')\ e^{i\mathbf{k}\cdot\mathbf{r}'}d^3\mathbf{r}'$, to calculate low energy cross section as $kR \ll 1$.

Next, we use the phase shift analysis. The s-wave of the low energy scattering of a central potential has a simple linear form $r\psi(r) = A(r-a)$ outside the potential core, as the incident wave number $k \to 0$. The parameter a is called the scattering length. It is known that the low energy cross section is simply given by $4\pi a^2$.

(b) Express the Schroedinger equation of the s-wave in different regions as $k \to 0$. Try the radial (reduced) wave function $u(r) \equiv r\psi(r) = r^{1+\alpha}$ when $r \le R$. Determine α and the scattering length a.

(c) Determine the low energy cross section (in terms of g and R). Compare your result with that obtained from the born approxiamtion. Explain the agreement in certain parametric region.

Problem 5. The lowest few wave functions of the usual hydrogen atom are listed below,

$$\Psi_{1S}(\mathbf{r}) = (\pi a_0^3)^{-\frac{1}{2}} e^{-r/a_0} \ , \quad \Psi_{2S}(\mathbf{r}) = (32\pi a_0^3)^{-\frac{1}{2}} (2 - r/a_0) e^{-\frac{r}{2a_0}} \ .$$

$$\Psi_{2P}^{m=0}(\mathbf{r}) = (32\pi a_0^3)^{-\frac{1}{2}} \cos\theta (r/a_0) e^{-\frac{r}{2a_0}} \ ,$$

where the Bohr radius $a_0 = \frac{\hbar}{m_e c}\frac{1}{\alpha}$, and the fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} = \frac{1}{137.036}$.

If there is a small short range modification to the coulomb attraction in the hydrogen atom

$$-\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r}\right) \longrightarrow -\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} + A\delta^{(3)}(\mathbf{x})\right) \ ,$$

the first order perturbation can be used to estimate its effect.

(a) Determine the energy shifts of states $2S$ and $2P$, also $E(2S) - E(2P)$ in terms of a_0, $\hbar c\alpha$, and A.

A muon is an elementary particle with the same charge as an electron but more massive, $m_\mu \approx 207 m_e$ ($m_e c^2 = 511000$ eV). When a muon ($m_\mu \approx 207 m_e$), is bounded to a proton, they form a *muonic hydrogen atom*. We can treat the muonic hydrogen just like the usual hydrogen. To further simplify the calculation, we ignore the proton motion in the system.

(b) Describe the lowest few energy states for this muonic hydrogen atom.

(c) The longest wavelength of the fist line H_α in the Balmer series of the usual hydrogen atom is 656.1 nm. What is the corresponding wavelength in the muonic hydrogen?

The muon, in comparison to the electron, can come much closer to the origin because of its large mass. Therefore the muonic hydrogen can better probe physics near the nucleus.

We know that the proton charge is screened by the positron-electron pair fluctuation, the potential near the center effectively picks up an additional short ranged piece, which is approximated by a delta function.

$$-\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r}\right) \longrightarrow -\frac{e^2}{4\pi\epsilon_0}\left(\frac{1}{r} + \frac{\alpha}{15\pi^2}\lambda_e^2 \delta^{(3)}(\mathbf{x})\right)$$

where the electron Compton wavelength $\lambda_e = \frac{h}{m_e c} = 0.002426$ nm.

(d) Determine analytically and numerically the small energy difference between the $2S$ and $2P$ states, due to this dominating screening effect in the muonic hydrogen.