Solution WYK

Wave Bouncing

We first list several lowest energy eigenstates:

$$\begin{aligned}
&\text{ground state:} \quad \psi_1(x) = \tfrac{1}{\sqrt{a}} \cos \tfrac{\pi x}{2a} \ , \quad E_1 = \frac{\hbar^2\pi^2}{8ma^2} \ , \\
&\text{1 st excited:} \quad \psi_2(x) = \tfrac{1}{\sqrt{a}} \sin \tfrac{\pi x}{a} \ , \quad E_2 = 2^2 E_1 \ , \\
&\text{2 nd excited:} \quad \psi_3(x) = \tfrac{1}{\sqrt{a}} \cos \tfrac{3\pi x}{2a} \ , \quad E_3 = 3^2 E_1 \ .
\end{aligned}$$

The initial state is a superposition of the 1st and 2nd excited states. We can easily find the normalization,

$$\psi(x,0) = C\sqrt{a}[3\psi_2(x) + 4\psi_3(x)] \ , \quad 1 = |C|^2 a(3^2+4^2) \ , C = \frac{1}{5\sqrt{a}} \ .$$

$$\psi(x,0) = \tfrac{1}{5}[3\psi_2(x) + 4\psi_3(x)] \ .$$

The probability is $\frac{9}{25}$ at the first excited state and $\frac{16}{26}$ at the next excited state. The mean energy is

$$\langle E \rangle = \left(\frac{9}{25} + \frac{16}{25}\frac{9}{4}\right)\frac{\hbar^2\pi^2}{2m_e a^2} = \frac{9\hbar^2\pi^2}{10m_e a^2}$$

The time dependent wave function is

$$\psi(x,t) = \tfrac{1}{5}[3\psi_2(x)e^{-iE_2t/\hbar} + 4\psi_3(x)e^{-iE_3t/\hbar}]$$

By symmetry, we know that

$$\langle\psi_2|x|\psi_2\rangle = 0 \ , \quad \langle\psi_3|x|\psi_3\rangle = 0 \ ,$$

So, the average position as a function of time is given by

$$\langle x \rangle_t = \int_{-a}^{+a} \psi(x,t)^* x \psi(x,t) dx$$

$$= \tfrac{1}{25}\left(9\langle\psi_2|x|\psi_2\rangle + 16\langle\psi_3|x|\psi_3\rangle + 12\langle\psi_3|x|\psi_2\rangle e^{-i(E_2-E_3)t/\hbar} + 12\langle\psi_2|x|\psi_3\rangle e^{i(E_2-E_3)t/\hbar}\right)$$

$$\langle x \rangle_t = \tfrac{24}{25}\langle\psi_3|x|\psi_2\rangle \cos\Omega t \qquad \text{with the frequency, } \Omega = (E_3 - E_2)/\hbar \ .$$

$$\langle\psi_3|x|\psi_2\rangle = \frac{1}{a}\int_{-a}^{a} x \sin\frac{\pi x}{a} \cos\frac{3\pi x}{2a} dx = \frac{1}{a}\left(\frac{2a}{\pi}\right)^2 2\int_0^{\pi/2} \theta \sin 2\theta \cos 3\theta d\theta = -\frac{96a}{25\pi^2}$$

$$\langle x \rangle_t = -\frac{2304}{625\pi^2} a \cos\Omega t$$

The period of the bouncing is $T = 2\pi/\Omega = h/(E_3 - E_2)$. It takes half period $T/2$ to flip the mean value of x to the opposite point.

Variation Principle

$$V(x,y) - V_0 = -V_0 e^{-\lambda^2 \mathbf{r}^2} \ , \quad V_0 \equiv \hbar^2\kappa_0^2/m \ , \quad \langle V - V_0\rangle_\beta = -V_0|C|^2 \int\int e^{-(\lambda^2+\beta^2)\mathbf{r}^2} d^2\mathbf{r}$$

$$\langle V - V_0\rangle_\beta = -V_0\pi|C|^2 \int_0^\infty e^{-(\lambda^2+\beta^2)r^2} 2rdr = -V_0\pi|C|^2/(\lambda^2+\beta^2)$$

The normalization condition gives similar result $|C|^2 = \beta^2/\pi$. So

$$\langle V - V_0\rangle_\beta = -V_0\beta^2/(\lambda^2+\beta^2) \ .$$

We go on to evalue the kinetic energy expectation value,

$$\left\langle \frac{\mathbf{p}^2}{2m} \right\rangle_\beta = \frac{\hbar^2}{2m}\int\int |\nabla\psi|^2 d^2\mathbf{r} = \frac{\hbar^2}{2m}\frac{\beta^2}{\pi}\int\int \beta^4\mathbf{r}^2 e^{-\beta^2r^2} d^2\mathbf{r}$$

$$= \frac{\hbar^2\beta^6}{2\pi m}\pi \int_0^\infty r^2 e^{-\beta^2r^2} 2rdr = \frac{\hbar^2\beta^6}{2m}\frac{1}{\beta^4} = \frac{\hbar^2\beta^2}{2m}$$

$$E(\beta^2) = \left\langle \frac{\mathbf{p}^2}{2m} \right\rangle_\beta + \langle V\rangle_\beta = \frac{\hbar^2\beta^2}{2m} + V_0 - V_0\frac{\beta^2}{\lambda^2+\beta^2} = \frac{\hbar^2\beta^2}{2m} + V_0\frac{\lambda^2}{\lambda^2+\beta^2}$$

$$E(\beta^2) = \frac{\hbar^2}{m}\left(\frac{\beta^2}{2} + \frac{\kappa_0^2\lambda^2}{\lambda^2+\beta^2}\right)$$

or $\boxed{A = \frac{1}{2},\ B = 1}$. In thye small $\lambda \to 0$ limit with the fixed product $\kappa_0\lambda = K^2$,

$$E(\beta^2) = \frac{\hbar^2}{m}\left(\frac{\beta^2}{2} + \frac{K^4}{\beta^2}\right) = \frac{\hbar^2}{2m}\left[\left(\beta - \frac{\sqrt{2}K^2}{\beta}\right)^2 + 2\sqrt{2}K^2\right]$$

Therefore, the minimum has a value $\boxed{E_{\rm min} = \sqrt{2}\hbar^2K^2/m \text{ at } \beta^2 = \sqrt{2}K^2\ .}$ In this special scenario, the potential near the origin is approxiamate by a quadratic form

$$V = (\hbar^2\kappa_0^2\lambda^2/m)r^2 = (\hbar^2K^4/m)r^2 \to \tfrac{1}{2}m\omega^2r^2 \ , \quad \omega^2 = 2\hbar^2K^4/m^2 \ .$$

$\boxed{\omega = \sqrt{2}\hbar K^2/m}$. The above minmum energy agrees with the zero-point energy $\hbar\omega$ in the 2-dim oscillator.

For the 1st excited levels, the wavefunctions has to be orthogonal to the ground state of a Gaussina function. The good choices are $\psi(x;\beta^2) = Dxe^{-\beta^2r^2}$ or $Dye^{-\beta^2r^2}$. The degenercy is TWO. The variation should give the same result as that from the oscillator, $E = 3\hbar\omega$.

Spins

We rewrite the Hamiltonian as $\mathcal{H}_0 = \frac{1}{2}A\left[(\mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3)^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2 - \mathbf{s}_3^2)^2\right]$. We note the property $\mathbf{s}_i^2 = \frac{3}{4}\hbar^2$. The total angular momentum operator, $\mathbf{J} = \mathbf{s}_1 + \mathbf{s}_2 + \mathbf{s}_3$, commutes with $\mathcal{H}_0$. The magnitude quantum number j labels the eigenvalue as $j(j+1)\hbar^2$.

$$\mathcal{H}_0 = \tfrac{1}{2}A\mathbf{J}^2 - \tfrac{9}{8}A\hbar^2$$

The vector space is a triple product of $\mathbf{\frac{1}{2}}$. The combination when pairing $\mathbf{s_1}$ and $\mathbf{s_2}$ is

$$\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}} = \left(\mathbf{\frac{1}{2}} \otimes \mathbf{\frac{1}{2}}\right) \otimes \mathbf{\frac{1}{2}} = (\mathbf{0} \oplus \mathbf{1}) \otimes \mathbf{\frac{1}{2}} = \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{1}{2}} \oplus \mathbf{\frac{3}{2}}\,.$$

Therefore, we have two doublets ($j = \frac{1}{2}$) and one quartet ($j = \frac{3}{2}$), counting a total of 8 states as expected. The two doublets come from different cominations, one as a singlet ($s_{12} = 0$) for $\mathbf{S}_{12} = \mathbf{s}_1 + \mathbf{s}_2$, and another as a triplet ($s_{12} = 1$).

Besides j and its magnetic quantum number m, we need s_{12} to label states. The energy eigenvalues $E_0(s_{12}, j)$ are

$$\begin{aligned}
E_0(0, \tfrac{1}{2}) &= \tfrac{1}{2}A\tfrac{3}{4}\hbar^2 - \tfrac{9}{8}A\hbar^2 &&= -\tfrac{3}{4}A\hbar^2 \\
E_0(1, \tfrac{1}{2}) &= &&= -\tfrac{3}{4}A\hbar^2 \\
E_0(1, \tfrac{3}{2}) &= \tfrac{1}{2}A\tfrac{15}{4}\hbar^2 - \tfrac{9}{8}A\hbar^2 &&= +\tfrac{3}{4}A\hbar^2
\end{aligned}$$

We notice that the energy trace $\sum E$ is zero as a good check.

$$\mathcal{H} = \mathcal{H}_0 + A\mathbf{s}_1 \cdot \mathbf{s}_2 = \mathcal{H}_0 + \tfrac{1}{2}A(\mathbf{S}_{12}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2) = \mathcal{H}_0 + \tfrac{1}{2}A(\mathbf{S}_{12}^2 - \tfrac{3}{2}\hbar^2)$$

The energy eigenvalues $E_(s12, j)$ are

$$E(s_{12}, j) = E_0(s_{12}, j) + \tfrac{1}{2}A\hbar^2[s_{12}(s_{12}+1) - \tfrac{3}{2}]$$

$$\begin{aligned}
E(0, \tfrac{1}{2}) &= -\tfrac{3}{4}A\hbar^2 - \tfrac{3}{4}A\hbar^2 &&= -\tfrac{3}{2}A\hbar^2 \\
E(1, \tfrac{1}{2}) &= -\tfrac{3}{4}A\hbar^2 + \tfrac{1}{4}A\hbar^2 &&= -\tfrac{1}{2}A\hbar^2 \\
E(1, \tfrac{3}{2}) &= +\tfrac{3}{4}A\hbar^2 + \tfrac{1}{4}A\hbar^2 &&= +A\hbar^2
\end{aligned}$$

We also note the zero energy trace as expected.

Scattering Length

Inside the potential range $r \leq R$, the s wave satisfies the simple Schroedinger equation as $k \to 0$,

$$-\frac{\hbar^2}{2m}\left(\frac{d^2}{dr^2}u - \frac{g}{r^2}\right) = 0 \ , \ \text{ for } r \leq R$$

Using the trial wave function, $u(r) = Ar^{1+\alpha}$, we find

$$(1+\alpha)\alpha = g \ , \quad \alpha = \tfrac{1}{2}(\sqrt{1+4g} - 1) \quad (\sim g \text{ as } g \ll 1) \ .$$

The other solution is singular at the origin and not acceptable. For a weak coupling of a small g, $\alpha \approx g$. Outside R, the free wave equation turns out to be $d^2u/dr^2 = 0$, and the solution is a straight line $u(r) = C(r - a)$. The continuity condition of the log-derivative at $r = R$ gives

$$\frac{1+\alpha}{R} = \frac{1}{R-a} \ , \qquad a = R\frac{\alpha}{1+\alpha} = \frac{\sqrt{1+4g}-1}{\sqrt{1+4g}+1} \quad (\sim g \text{ as } g \ll 1) \ .$$

The cross section at the low energy ($k \to 0$) is $\sigma \approx 4\pi a^2$. In the weak $g \ll 1$ limit, it is $\boxed{4\pi R^2 g^2}$.

On the other hand, the Bohr approximation gives

$$f_{\mathbf{k}}(\hat{\mathbf{k}'}) \xrightarrow{k\to 0} -\frac{m}{2\pi\hbar^2}\int V(\mathbf{r}')d^3\mathbf{r}' = -\frac{m}{2\pi\hbar^2}\frac{\hbar^2}{2m}\int_0^R \frac{g}{r'^2}4\pi r'^2 dr' = -gR$$

$$d\sigma/d\Omega = |f_{\mathbf{k}}(\hat{\mathbf{k}'})|^2 \to g^2R^2$$

We obtain the same cross section as before in the weak g coupling as the perturbation is valid.

Muonic Hydrogen Atom

If we ignore the proton motion, all energy scale is enchaced by the factor 207 which is the mass ratio m_μ/m_e. The few lowest energy values in the usual hydrogen atoms
$E_1^e = -13.6$ eV , $E_2^e = -3.4$ eV, $E_3^e = -1.5$ eV,
are enhanced respectively in the muonic hydrogen,
$E_1^\mu = -13.6 \times 207$ eV $= 2815$ eV,
$E_2^\mu = -3.4 \times 207$ eV $= 704$ eV,
$E_3^\mu = -1.5 \times 207$ eV $=313$ eV.
The Balmer line, 656.1 nm, in the usual hydrogen is reduced to 656.1 nm/207 = 3.17 nm.

It is important to know that the vacuum polarization is short ranged, as given the delta fucntion at the origin, and it only affects the S wave. Therefore $\boxed{\Delta E(2P) = 0}$.

$$\Delta E(2S) = -\frac{e^2}{4\pi\epsilon_0}\frac{\alpha}{15\pi^2}\frac{4\pi^2\hbar^2}{m_e^2c^2}|\psi_{2S}(0)|^2 = -\alpha\hbar c\frac{\alpha}{15\pi^2}\frac{4\pi^2\hbar^2}{m_e^2c^2}\left(\frac{1}{8\pi a_0^3}\right)$$

$$\Delta E(2S) = -\frac{\alpha^2\hbar^3}{30\pi m_e^2 c}\left(\frac{\alpha m c}{\hbar}\right)^3 = -\frac{\alpha^5}{30\pi}\left(\frac{m}{m_e}\right)^3 m_e c^2 \ .$$

Here m is m_e in the usual electronic hydrogen, or m_μ in the muonic one. The numerical value for the usual hydrogen is $\Delta E(e, 2S) = -1.12 \times 10^{-7}$ eV $= -27$ MHz h, that is part of the Lamb shift, but overwhelmed by other effects. However, in the muonic hydrogen, the screening is enchanced by $m_\mu/m_e)^3$,

$$\Delta E(\mu, 2S) = -1.12 \times 10^{-7}\ \text{eV} \ \times (207)^3 \approx -1\ \text{eV}$$

To conclude, the 2P level is not affected, but 2S becomes lower by about 1 eV because the muon feel more bare proton charge at the origin.
$\boxed{E(\mu, 2S) - E(\mu, 2P) = -1\ \text{eV}}$.