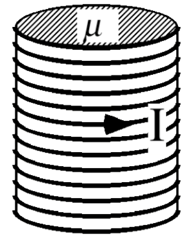
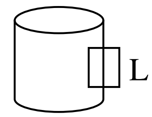
e&m possibilities 2006 (solutions)



An infinitely long cylinder of linear magnetic material of permeability μ is wrapped with a wire (forming an infinite solenoid of radius R wrapped around the cylinder). The wire carries a current I and has N loops per unit length. Ignore the magnetic properties of the wire.

(a) Calculate the field $\vec{H}$ as a function of s, the radial distance from the cylinder axis, for $s > R$ and $s < R$.



First, find **H** using $\oint \mathbf{H} \cdot d\mathbf{l} = I_{free}$, so HL=INL $\Rightarrow \mathbf{H} = \begin{cases} IN\hat{z} \text{ inside, } s < R \\ 0 \text{ outside, } s > R \end{cases}$

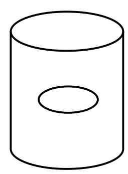
Justification for **H**=0 outside is that **H** depends only on I_{free} and $\mathbf{B}_{out} = 0$ for solenoid without core.

(a) Calculate the magnetic vector potential $\vec{A}$ as a function of s, the radial distance from the cylinder axis, for $s > R$ and $s < R$.

$$\mathbf{B} = \mu\mathbf{H} = \begin{cases} \mu IN\hat{z} \text{ inside, } s < R \\ 0 \text{ outside, } s > R \end{cases}$$

Calculate **A** from $\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a}$

By symmetry choose integration path shown in figure.



$$s < R : A2\pi s = B\pi s^2 \Rightarrow \mathbf{A} = \frac{B}{2} s\hat{\phi}$$

$$s > R : A2\pi s = B\pi R^2 \Rightarrow \mathbf{A} = \frac{BR^2}{2s}\hat{\phi}$$

(a) Calculate the surface and volume bound currents in the magnetic cylinder. Note that these currents are sometimes referred to as magnetization currents.

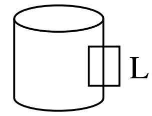
$\mathbf{M} = \chi_m \mathbf{H}$, where χ_m is given by $\mu = \mu_o(1 + \chi_m)$, so $\mathbf{M} = \chi_m IN\hat{z}$

Volume bound current, $\mathbf{J}_b = \nabla \times \mathbf{M} = 0$, since M is constant.

Surface bound current, $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \hat{\mathbf{s}}$, so $\mathbf{K}_b = \chi_m IN\hat{\phi}$, collinear with I_{free}

(a) Use the free and bound currents to calculate $\vec{B}$ as a function of s, the radial distance from the cylinder axis, for $s > R$ and $s < R$.

Find $\mathbf{B}$ using $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_o I_{total} = \mu_o I_{free} + \mu_o I_{bound}$ (use path shown in figure)



BL=$[\mu_o NI + \mu_o \chi_m NI]L = [\mu_o NI + \mu_o(\mu / \mu_o - 1)NI]L = \mu NIL$, so $\mathbf{B} = \mu NI\hat{z}$ as in part (b) for $s < R$.

Since I_{bound} is collinear with I_{free}, the symmetry is that of an infinitely long solenoid without a core, which implies that $\mathbf{B} = 0$ for $s > R$.

A (non-conducting) solid sphere of radius R carries a charge density $\rho(r) = kr^2$ (where k is a constant).
Find the electric field at a distance r such that
(a) $r \geq R$

$\oint \mathbf{E} \cdot d\mathbf{a} = Q_{enc} / \varepsilon_o$ By symmetry of charge distribution $\mathbf{E} = E_r(r)\,\hat{r}$; also d$\mathbf{a} = da\,\hat{r}$

$\oint \mathbf{E} \cdot d\mathbf{a} = E_r(r)\, 4\pi r^2$

Since ρ depends on r, $Q_{enc} = \int \rho \, d\tau$

$$E_r(r) = \frac{1}{4\pi r^2 \varepsilon_o} \int_0^R kr^2 4\pi r^2 \, dr, \text{ or } \mathbf{E} = \frac{kR^5}{5\varepsilon_o r^2}\hat{r}$$

(b) $0 \leq r \leq R$

Similar to part (a): $E_r(r) = \dfrac{1}{4\pi r^2 \varepsilon_o} \displaystyle\int_0^r kr^2 4\pi r^2 \, dr$, or $\mathbf{E} = \dfrac{kr^3}{5\varepsilon_o}\hat{r}$

(c) Show that your answers to parts (a) and (b) are consistent with the boundary conditions on the electric field.

$\mathbf{E}_{above} - \mathbf{E}_{below} = \dfrac{\sigma}{\varepsilon_o}\hat{n}$, where "above" is at $r = R^+$ and "below" is at $r = R^-$, $\hat{n} = \hat{r}$ and $\sigma = 0$.

$$\left.\frac{kR^5}{5\varepsilon_o r^2}\right|_{r=R^+} - \left.\frac{kr^3}{5\varepsilon_o}\right|_{r=R^-} = 0 \quad \text{for } \hat{r} \text{ component } (E_\perp).$$

Also $E_\phi = E_\theta = 0$, so $E_{||}$ is continuous across the surface.

(d) Calculate the electric potential for all r. Let $V = 0$ at $r = \infty$.

$V = -\int \mathbf{E} \cdot d\mathbf{l}$ Take path l to be in $\hat{r}$ direction and

for $r \geq R$ $\quad V = -\displaystyle\int_\infty^r \frac{kR^5}{5\varepsilon_o r^2}\hat{r} \cdot (\hat{r}\, dr) = \frac{kR^5}{5\varepsilon_o r}$

for $r \leq R$ $\quad V = -\displaystyle\int_\infty^R \frac{kR^5}{5\varepsilon_o r^2}\hat{r} \cdot (\hat{r}\, dr) - \int_R^r \frac{kr^3}{5\varepsilon_o}\hat{r} \cdot (\hat{r}\, dr) = \frac{kR^4}{5\varepsilon_o} - \frac{k}{5\varepsilon_o}\left(\frac{r^4}{4} - \frac{R^4}{4}\right) = \frac{k}{20\varepsilon_o}\left(5R^4 - r^4\right)$

(e) Find the work required to assemble this charge distribution.

Several ways to calculate the energy, including the following that use previous results of this problem.

$$W=\frac{1}{2}\int \rho V d\tau = \frac{k}{40\varepsilon_o}\int_0^R kr^2\left[5R^4 - r^4\right]4\pi r^2\, dr = \frac{4\pi k^2}{45\varepsilon_o}R^9$$

OR, $W = \frac{\varepsilon_o}{2}\int E^2 d\tau = \frac{\varepsilon_o}{2}\int_R^\infty \frac{kR^5}{5\varepsilon_o r^2}\, d\tau + \frac{\varepsilon_o}{2}\int_0^R \frac{kr^3}{5\varepsilon_o}\, d\tau$ which gives the same answer.

(f) If the *non*-conducting solid sphere was replaced by a *conducting* solid sphere with the same total charge, how does that change your answer to parts (a), (b), and (c)?

(a) $\mathbf{E}$ is unchanged

(b) $\mathbf{E} = 0$ inside the conducting sphere

(c) All charges move to the surface. The surface charge density σ is given by

$$4\pi R^2\sigma = \int_0^R kr^2 d\tau = \int_0^R kr^2(4\pi r^2 dr) \Rightarrow \sigma = \frac{kR^3}{5}$$

So, the boundary conditions on $\mathbf{E}$ are given by

$$E_{\perp,above} - E_{\perp below} = \frac{\sigma}{\varepsilon_o} = \frac{kR^3}{5\varepsilon_o}$$

$\left.\frac{kR^5}{5\varepsilon_o r^2}\right|_{r=R^+} - 0 = \frac{kR^3}{5\varepsilon_o}$, as expected. As before, $E_\phi = E_\theta = 0$, so $E_{||}$ is continuous across the surface.

Consider a sphere of radius R that is uniformly polarized along the z–axis, such that $\mathbf{P} = P\hat{z}$.
(a) Determine the bound charge distribution (sometimes called the polarization charge).
(a) *Derive* the electric potential produced by this polarized sphere.

Since the polarization is constant, the volume bound charge is zero. The surface bound charge is given by $\sigma_b = \mathbf{P}\cdot\mathbf{n} = P\cos\theta$, where θ is the polar angle in spherical coordinates. There are several ways to do this problem, one is to use Coulomb's law, the other is to guess the answer (but the problem asks for a derivation), and the hint suggests solving for the potential using Laplace's equation by using the following solution: $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

For $r \le R$, the lack of charge inside leads to $V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l\right)P_l(\cos\theta)$

For $r \ge R$, the boundary condition $V(\infty) = 0$ leads to $V(r,\theta) = \sum_{l=0}^{\infty}\left(\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$

The potential is continuous on the boundary, $r = R$: $\sum_{l=0}^{\infty}\left(A_l R^l\right)P_l(\cos\theta) = \sum_{l=0}^{\infty}\left(\frac{B_l}{R^{l+1}}\right)P_l(\cos\theta)$

Orthogonality of the Legendre polynomials leads to $A_l R^l = \frac{B_l}{R^{l+1}}$, or $B_l = A_l R^{2l+1}$.

The boundary condition on the normal derivatives of the potential: $\left.\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\right|_{r=R} = -\frac{\sigma}{\varepsilon_o}$

$$-\sum_{l=0}^{\infty}(l+1)\left(\frac{B_l}{R^{l+2}}\right)P_l(\cos\theta) - \sum_{l=0}^{\infty} l\left(A_l R^{l-1}\right)P_l(\cos\theta) = -\frac{P\cos\theta}{\varepsilon_o}$$

Using $B_l = A_l R^{2l+1}$, yields $\sum_{l=0}^{\infty}(2l+1)\left(A_l R^{l-1}\right)P_l(\cos\theta) = \frac{P\cos\theta}{\varepsilon_o}$

Fourier analysis: $\sum_{l=0}^{\infty}(2l+1)\left(A_l R^{l-1}\right)\int_0^{\pi} P_l(\cos\theta)P_n(\cos\theta)\, d(\cos\theta) = \frac{P}{\varepsilon_o}\int_0^{\pi}\cos\theta\, P_n(\cos\theta)\, d(\cos\theta)$

Noting that $\cos\theta = P_1(\cos\theta)$ and that only the $n = l = 1$ term is non-zero
(This could also be done using the orthonormality condition, to be given in equation sheet)

Only $l = 1$ term survives: $A_1 = \frac{P}{3\varepsilon_o}$, so $V(r,\theta) = \begin{cases} \frac{P}{3\varepsilon_o} r\cos\theta & \text{for } r \le R \\ \frac{P}{3\varepsilon_o}\frac{R^3}{r^2}\cos\theta & \text{for } r \ge R \end{cases}$

When an electron is injected at right angles to a steady uniform magnetic field, ***B***, it initially executes circular motion about the magnetic field lines.

(a) Assume that the electron is non-relativistic with a velocity $v << c$. Determine the power radiated in terms of only the electron's velocity, the angular frequency of orbiting, ω_c, and any other necessary constants. To determine the cyclotron frequency, ω_c, you may assume that the velocity is nearly constant for each (nearly) circular revolution.

The radiated power depends upon the square of the acceleration, as given by the Larmor formula:

$$P = \frac{q^2 a^2}{6\pi\varepsilon_o c^3}$$

The long way to find a: $\mathbf{F} = q\mathbf{v} \times \mathbf{B} = m\mathbf{a} \Rightarrow qvb = mv^2 / R$, so $R = mv / qB$.

So $a = v^2 / R = qvB / m$.

Since the orbiting or cyclotron frequency ω_c is given by $v = \omega_c R$, we have $a = \omega_c v$.

The short way to find a: $a = v^2 / R$ and $v = \omega_c R$, so $a = \omega_c v$.

So, the power radiated per cycle is

$$P_{cycle} = \frac{q^2 (\omega_c v)^2}{6\pi\varepsilon_o c^3}$$

(b) Due to the power radiated in part (a), energy is continually lost by the electron and it slows down. First, derive an expression for the total energy of the electron (kinetic plus potential) by taking the potential energy to be $-\mu \cdot \mathbf{B}$, where μ is the magnetic moment of a circular loop of current due to the electron. This expression for the total energy can be written simply in terms of only the electron mass m and its velocity v.
Next, derive an expression for the time that it takes the energy to fall to 1/e of its initial value.

First, the total energy is the sum of the sum of the kinetic, $\frac{1}{2}mv^2$, and potential, $-\mu \cdot \mathbf{B}$, energies.

If the magnetic moment is taken as the moment of the current loop due to the electron, then

$\mu = IA$, where $A = \pi R^2$ and $I = \frac{qv}{2\pi R}$, so $\mu = \frac{qv}{2\pi R}\pi R^2 = \frac{qvR}{2} = \frac{mv^2}{2B}$,

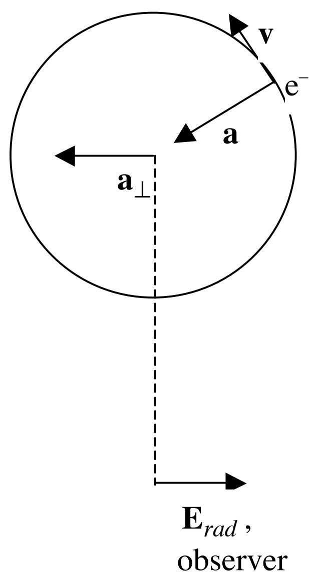
so the potential energy is $\frac{1}{2}mv^2$ (note that μ and $\mathbf{B}$ are in opposite directions).

Second, $P(t) = -\frac{dU}{dt} = \frac{q^2(\omega_c v)^2}{6\pi\varepsilon_o c^3} = \frac{q^2\omega_c^2 mv^2}{6\pi\varepsilon_o c^3 m} = \frac{q^2\omega_c^2 U}{6\pi\varepsilon_o c^3 m}$

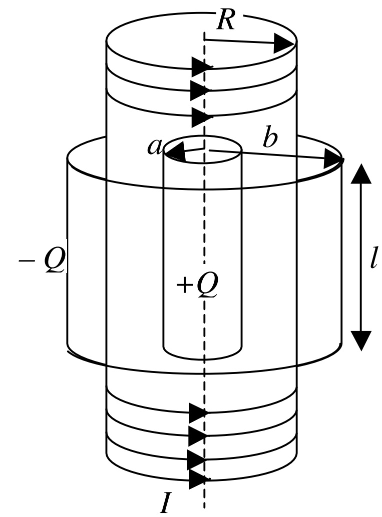
So, $U = U_o e^{-t/\tau}$, where $\tau = \frac{6\pi\varepsilon_o c^3 m}{q^2\omega_c^2} = \frac{6\pi\varepsilon_o c^3 m^3}{q^4 B^2}$, using $\omega_c = qB / m$

(c) Describe the direction of polarization of the electromagnetic radiation seen by an observer within the plane determined by the path of the electron.

The direction of the radiated electric field is opposite to that of the component of the acceleration perpendicular to observer's line of sight, $\mathbf{E}_{rad} \propto -\mathbf{a}_{\perp}$.



Consider a very long solenoid with radius R, n turns per length, and current I_o. Coaxial with the solenoid are two long cylindrical shells of length l. One of the shells is inside the solenoid, of radius a, and carries a total charge $+Q$ that is uniformly distributed over the shell. The other shell is outside the solenoid, of radius b, and carries a total charge $-Q$ that is uniformly distributed over the shell. So, $a < R < b$ and $l >> a$ and $l >> b$ (note that the figure does not have these relative dimensions correct – for the purpose of calculating electric and magnetic fields the solenoid and cylinders can be considered infinitely long). The charged cylinders are free to rotate about their axis.



a) The current in the solenoid is gradually reduced to zero. Calculate the final angular momentum of the two charged cylinders.

When the current is turned off the changing **B** induces a circumferential **E**.

$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}$. Since $B = \mu_o n I$:

$$\mathbf{E} = \begin{cases} -\frac{1}{2}\mu_o n \frac{dI}{dt} s\hat{\phi} & \text{for } s < R \\ -\frac{1}{2}\mu_o n \frac{dI}{dt}\frac{R^2}{s}\hat{\phi} & \text{for } s > R \end{cases}$$

This electric field will exert a torque **N** on the charged cylinders

$$\mathbf{N} = \begin{cases} \mathbf{r}\times(Q\mathbf{E}) = -\frac{1}{2}\mu_o n Q a^2 \frac{dI}{dt}\hat{z} \text{ for the inner cylinder} \\ \mathbf{r}\times(-Q\mathbf{E}) = \frac{1}{2}\mu_o n Q R^2 \frac{dI}{dt}\hat{z} \text{ for the outer cylinder} \end{cases}$$

The angular momentum imparted to the cylinders is given by $\int \mathbf{N}\, dt = ()\int_{I_o}^{0} \frac{dI}{dt}\, dt = -()I_o$

$$\mathbf{L} = \begin{cases} \frac{1}{2}\mu_o n Q a^2 I_o \hat{z} \text{ for the inner cylinder} \\ -\frac{1}{2}\mu_o n Q R^2 I_o \hat{z} \text{ for the outer cylinder} \end{cases}$$

$$\mathbf{L}_{total} = \frac{1}{2}\mu_o n Q(a^2 - R^2) I_o \hat{z}$$

b) Demonstrate that angular momentum is conserved by considering the angular momentum before and after the current is turned off. You can assume that the magnetic field is negligible after the current in the solenoid is turned off.

The final angular momentum of the cylinders comes from the angular momentum in the original electric and magnetic fields.

The electric field due to the charged cylinders can be derived from Gauss's law in integral form $\mathbf{E} = \dfrac{Q}{2\pi\varepsilon_o l s}\hat{\mathbf{s}}$ for $a < s < b$. The initial magnetic field inside the solenoid is $\mathbf{B} = \mu_o n I_o \hat{\mathbf{z}}$.

The angular momentum density is given by $\mathbf{r} \times (\varepsilon_o \mathbf{E} \times \mathbf{B}) = \dfrac{Q\mu_o n I_o}{2\pi l}\hat{\mathbf{r}} \times (\hat{\mathbf{s}} \times \hat{\mathbf{z}}) = -\dfrac{Q\mu_o n I_o}{2\pi l}\hat{\phi}$

The total angular momentum in the fields is given by this density times the volume occupied by the region containing both fields, so $\mathbf{L}_{em} = -\dfrac{Q\mu_o n I_o}{2}(R^2 - a^2)\hat{\phi}$. This is the same as the total $\mathbf{L}$ transferred to the cylinders.