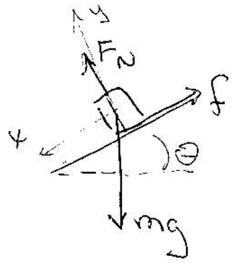
CLASSICAL MECHANICS F06 SOLUTIONS

1.



$$\Sigma \vec{F}_x := mg\sin\theta - \vec{f} = ma$$
$$= mg\sin\theta - \mu_s \vec{F}_N = mg\sin\theta - \mu_s mg\cos\theta$$

$$\Sigma \vec{F}_y = 0 = \vec{F}_N - mg\cos\theta$$

so $a = g(\sin\theta - \mu_s\cos\theta)$

a. Max θ found where $a = 0$.

$$\tan\theta = \frac{\mu_s}{g}$$

b. $a_{max} = g(\sin\theta - \mu_s\cos\theta)$

c. ~~After~~ the truck goes over a bump
F_N decreases since $a_y < 0$ for a moment.
If F_N decreases so does the frictional force

d. The box slides forward into the cab
Since its sliding $f = \mu_k F_N < \mu_s F_N$. $a_{box} > a_{truck}$

$a_{box} = g(\sin\theta - \mu_k\cos\theta)$; $a_t = g(\sin\theta - \mu_s\cos\theta)$

$\Delta x_b = \frac{1}{2}a_b t^2 + v_0 t$ $\Delta x_t = \frac{1}{2}a_t t^2 + v_0 t$

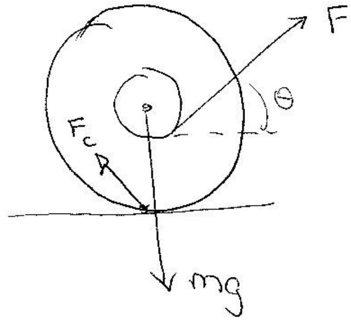
Motion of box with respect to truck is

$$\Delta x = \Delta x_b - \Delta x_t = \frac{t^2}{2}g(-\mu_k g\cos\theta + \mu_s\cos\theta)$$

$$L = \frac{gt^2}{2}\cos\theta(\mu_s - \mu_k) \quad \text{so} \quad t^2 = \frac{2L}{g\cos\theta}\frac{1}{0.1}$$

$$t = \frac{20L}{g\cos\theta}$$

2.



$$y: \quad \sum F_y = F_{cy} + F\sin\theta - mg = 0$$

$$x: \quad \sum F_y = -F_{cx} + F\cos\theta = MR\frac{d\omega}{dt}$$

$$\tau \text{ about CM} \quad \sum \vec{\tau} = +F_{cx}R - Fr = I_0\frac{d\omega}{dt}$$

$$F_{cx} = F\cos\theta - MR\frac{d\omega}{dt}$$

$$\left(F\cos\theta - MR\frac{d\omega}{dt}\right)R - Fr = I_0\frac{d\omega}{dt}$$

$$\frac{d\omega}{dt} = \frac{FR\cos\theta - Fr}{I_0 + MR^2}$$

Forward motion occurs when $\frac{d\omega}{dt} > 0$ if

$$FR\cos\theta > Fr \qquad \cos\theta > \frac{r}{R}$$

Backward motion occurs when $\cos\theta < \frac{r}{R}$

so. $\cos\theta_c = \frac{r}{R}$

3. Use $l = mr^2 \frac{d\theta}{dt}$ to relate $l\,dt = mr^2\,d\theta$ variations

$$\frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta} \quad \text{and} \quad \frac{d^2}{dt^2} = \frac{l}{mr^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{d}{d\theta}\right)$$

notice $\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d(1/r)}{d\theta} = -\frac{du}{d\theta}$

Lagrange equation $m\ddot{r} - \frac{l^2}{mr^3} = f(r)$ becomes

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} = f(r)$$

$$\frac{l^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = f(u)$$

b. $u = \frac{1}{r} = \frac{1}{r_0}e^{-k\theta}$

$$\frac{du}{d\theta} = -\frac{k}{r_0}e^{-k\theta}$$

$$\frac{d^2u}{d\theta^2} = +\frac{k^2}{r_0}e^{-k\theta} = k^2u$$

$\Rightarrow$

$$k^2u + u = -\frac{m}{l^2u^2}f(1/u)$$

$$f\left(\frac{1}{u}\right) = \frac{-l^2(k^2+1)u^3}{m}$$

$$f(r) = -\frac{l^2(k^2+1)}{mr^3}$$

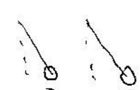
c. $l = mr^2\frac{d\theta}{dt}$ so $dt = \frac{mr^2}{l}d\theta = \frac{m}{l}r_0^2e^{2k\theta}\,d\theta$

$$t = \frac{mr_0^2}{l}\frac{1}{2k}e^{2k\theta}$$

$$\frac{2klt}{mr_0^2} = e^{2k\theta} \quad \Rightarrow \quad 2k\theta = \ln\left(\frac{2klt}{mr_0^2}\right)$$

$$\theta(t) = \frac{1}{2k}\ln\left(\frac{2klt}{mr_0^2}\right)$$

$x_- = 0 \Rightarrow x_1 = x_2$ at ω_1



4.a. For small displacements

$$m\ddot{x}_1 = -\frac{mg}{L}x_1 - k(x_1 - x_2)$$

$$2m\ddot{x}_2 = -\frac{2mg}{L}x_2 + k(x_1 - x_2)$$

Solutions are of type $x_1 = A e^{+i\omega t}$; $x_2 = B e^{+i\omega t}$

$$\left(m\omega^2 - \frac{mg}{L} - k\right)A + kB = 0$$

$$\left(2m\omega^2 - \frac{2mg}{L} - k\right)B + kA = 0 \qquad \text{let } C = m\omega^2 - \frac{mg}{L}$$

$$\begin{vmatrix} C-k & k \\ k & 2C-k \end{vmatrix} = 0$$

$$(C-k)(2C-k) - k^2 = 0$$

$$2C^2 - (Ck + 2Ck) = 0$$

$$C(2C - 3k) = 0$$

Two solutions:

$$m\omega_1^2 - mg/L = 0 \qquad 2m\omega_2^2 - 2mg/L - 3k = 0$$

$$\omega_1 = \pm\sqrt{g/L} \qquad \omega_2 = \pm\sqrt{\frac{3k}{2m} + g/L}$$

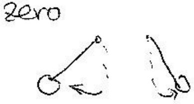
for $A = B$ $\qquad$ for $A = -2B$

b. General solution:

$$\left.\begin{aligned} x_1 &= A_1 e^{i\omega_1 t} + A_2 e^{i\omega_2 t} \\ x_2 &= A_1 e^{i\omega_1 t} - \frac{A_2}{2} e^{i\omega_2 t} \end{aligned}\right\} \rightarrow \begin{aligned} x_+ &= x_1 + 2x_2 = 3A_1 e^{i\omega_1 t} \\ x_- &= x_1 - x_2 = \tfrac{3}{2}A_2 e^{i\omega_2 t} \end{aligned}$$

c. Normal modes: set one solution to zero

$x_+ = 0 \Rightarrow x_1 = -2x_2$ at ω_2



$x_- = 0 \Rightarrow x_1 = x_2$ at ω_1

5a. Let r be the distance to bead

$x = r\sin\theta\cos\omega t$

$y = r\sin\theta\sin\omega t$

$z = A - r\cos\theta$

$$KE = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$= \frac{1}{2}m\left[(\dot{r}\sin\theta\cos\omega t - \omega r\sin\theta\sin\omega t)^2 + (\dot{r}\sin\theta\sin\omega t + \omega r\sin\theta\cos\omega t)^2 + (-\dot{r}\cos\theta)^2\right]$$

$$= \frac{1}{2}m\left[\dot{r}^2\sin^2\theta + \omega^2 r^2\sin^2\theta + \dot{r}^2\cos^2\theta\right]$$

$$= \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2\sin^2\theta)$$

V assuming $V=0$ at xy plane

$V = mg(A - r\cos\theta)$

$L = T - V = \frac{1}{2}m(\dot{r}^2 + \omega^2 r^2\sin^2\theta) - mg(A - r\cos\theta)$

b. $\frac{\partial L}{\partial r} = m\omega^2 r\sin^2\theta + mg\cos\theta$ and $\frac{\partial L}{\partial \dot{r}} = m\dot{r}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$ so

$m\ddot{r} - m\omega^2 r\sin^2\theta - mg\cos\theta = 0$

$\ddot{r} - \omega^2\sin^2\theta\, r = g\cos\theta$

c. General solution if $g\cos\theta$ set to zero: $a_1 e^{\omega\sin\theta t} + a_2 e^{-\omega\sin\theta t}$

$g\cos\theta$ is a constant so the particular solution is $\frac{-g\cos\theta}{\omega^2\sin^2\theta}$

$$\therefore\ r = a_1 e^{\omega\sin\theta t} + a_2 e^{-\omega\sin\theta t} - \frac{g\cos\theta}{\omega^2\sin^2\theta}$$