

1. (a) Use Boltzmann distribution

$$\rho(z) = Ae^{-\beta mgz}$$

normalization determined by

$$\int d^3r \rho(z) = N = AL^2 \int_0^L dz e^{-\beta mgz} = \frac{AL^2}{\beta mg} (1 - e^{-\beta mgL})$$

$$\rho(z) = \frac{N\beta mgL}{V(1 - e^{-\beta mgL})} e^{-\beta mgz}$$

(b) Probably easiest to use canonical partition func

$$Z = \frac{1}{N!} \left(\left[\frac{2\pi mk_B T}{h^2} \right]^{3/2} L^3 \int_0^L \frac{dz}{L} e^{-\beta mgz} \right)^N$$

Take limit $mgL/k_B T \ll 1$ right at the start, then integral is

$$\begin{aligned} \int_0^L \frac{dz}{L} e^{-\beta mgz} &= \int_0^L \frac{dz}{L} \left(1 - \frac{1}{2!} \beta mgz + \frac{1}{3!} (\beta mgz)^2 + \dots \right) \\ &= 1 - \beta mgL + (\beta mgL)^2 - \frac{1}{2} (\beta mgL)^3 + \dots \end{aligned}$$

so

$$Z = e^{-\beta F} = \left(V \left[\frac{2\pi mk_B T}{h^2} \right]^{3/2} \left[1 - \frac{1}{2} \beta mgL + \frac{1}{3!} (\beta mgL)^2 + \dots \right] \right)^N$$

so

$$-F = Nk_B T \left[\ln V - \ln N + 1 + \frac{3}{2} \ln \frac{2\pi mk_B T}{h^2} + \ln \left(1 - \frac{1}{2} \beta mgL + \frac{1}{3!} (\beta mgL)^2 + \dots \right) \right]$$

need to expand log to 2nd order in βmgL ,

$$-F = Nk_B T \left[\ln \frac{V}{N} + 1 + \frac{3}{2} \ln \frac{2\pi mk_B T}{h^2} + \frac{1}{2} \beta mgL + \frac{1}{12} (\beta mgL)^2 + \dots \right]$$

finally use $S = -(\partial F / \partial T)_{N,V}$

$$S(N, T, V, g) = Nk_B \left[\ln \frac{V}{N} + \frac{5}{2} + \frac{3}{2} \ln \frac{2\pi mk_B T}{h^2} - \frac{1}{12} \left(\frac{mgL}{k_B T} \right)^2 \right]$$

note sign of g doesn't matter, and S goes down as g goes up, since fewer positional states are occupied; the gravitational field orders the gas

(c) easy to see that if temp held constant, sending gas into space raises its disorder

$$\Delta S = S(N, T, V, 0) - S(N, T, V, g) = \frac{Nk_B}{12} \left(\frac{mgL}{k_B T} \right)^2 + \dots$$

so entropy goes up

(d) now send gas up without heat transfer; $\delta S = 0$, or $S(N, T, V, g) = S(N, T_f, V, 0)$, giving us

$$\frac{3}{2} \ln T - \frac{1}{12} \left(\frac{mgL}{k_B T} \right)^2 = \frac{3}{2} \ln T_f$$

or

$$\left(\frac{T_f}{T} \right)^{3/2} = \exp \left[-\frac{1}{12} \left(\frac{mgL}{k_B T} \right)^2 \right]$$

or

$$T_f = T \exp \left[-\frac{1}{18} \left(\frac{mgL}{k_B T} \right)^2 + \dots \right] = T \left[1 - \frac{1}{18} \left(\frac{mgL}{k_B T} \right)^2 + \dots \right]$$

so temperature goes down when gas is launched into space without heat transfer.

2. (a) Since elements are independent, we can compute the partition function for one element, $Z_1 = 1 + e^{-\beta\epsilon}$, and then compute $Z_N = (1 + e^{-\beta\epsilon})^N$. Since the energy is just proportional to the number of elements in state b , we can compute

$$\langle N_B \rangle = -\frac{\partial \ln Z}{\partial \beta\epsilon} = \frac{N}{e^{\beta\epsilon} + 1}$$

Note that $\langle N_A \rangle = N - \langle N_B \rangle$, so

$$\langle L \rangle = a \langle N_A \rangle + b \langle N_B \rangle = Na + \frac{N(b-a)}{e^{\beta\epsilon} + 1}$$

$$(b) \langle L^2 \rangle = \langle (aN_A + bN_B)^2 \rangle = \langle (aN + (b-a)N_B)^2 \rangle$$

$$= a^2N^2 + 2a(b-a)N \langle N_B \rangle + (b-a)^2 \langle N_B^2 \rangle$$

$$\langle L \rangle^2 = [aN + (b-a) \langle N_B \rangle]^2 = a^2N^2 + (b-a)^2 \langle N_B \rangle^2 + 2a(b-a)N \langle N_B \rangle$$

so

$$\langle L^2 \rangle - \langle L \rangle^2 = (b-a)^2 \left[\langle N_B^2 \rangle - \langle N_B \rangle^2 \right]$$

and

$$\langle N_B^2 \rangle - \langle N_B \rangle^2 = \frac{\partial^2}{\partial (\beta\epsilon)^2} \ln Z = \frac{Ne^{\beta\epsilon}}{(e^{\beta\epsilon} + 1)^2}$$

giving RMS fluctuation

$$\sqrt{\langle L^2 \rangle - \langle L \rangle^2} = \frac{\sqrt{N}(b-a)}{e^{\beta\epsilon/2} + e^{-\beta\epsilon/2}}$$

(c) If we force length to be L we force $N_B = (L - Na)/(b - a)$. The energy is $E = N_B\epsilon$, and the entropy is just the log of the number of ways to choose N_B of the monomers to be in state b , $S = k_B \ln \binom{N}{N_B}$. So,

$$S = k_B \ln \frac{N!}{N_B!(N - N_B)!}$$

which can be written in the large N and N_B limit as

$$S = k_B \left[N_B \ln \frac{N}{N_B} + (N - N_B) \ln \frac{N}{N - N_B} \right]$$

so

$$F = E - ST = N_B\epsilon + k_B T \left[N_B \ln \frac{N_B}{N} + (N - N_B) \ln \frac{N - N_B}{N} \right]$$

where $N_B/N = (L - Na)/[N(b - a)]$ and where $1 - N_B/N = (Nb - L)/[N(b - a)]$.

(d) Force is given by L derivative of free energy at fixed temperature

$$f = \left(\frac{\partial F}{\partial L} \right)_T = \left(\frac{\partial N_B}{\partial L} \right) \left(\frac{\partial F}{\partial N_B} \right)$$

$$f = (\epsilon + k_B T [\ln N_B/N - \ln(1 - N_B/N)]) / (b - a)$$

or

$$f = \frac{\epsilon + k_B T \ln \frac{L - Na}{Nb - L}}{b - a}$$

Note force diverges to infinite tension if $L \rightarrow Nb$; also force diverges to infinite compression if $L \rightarrow Na$.

3. (a) At 10^7 K we estimate KE per neutron

$$k_B T = 1.38 \times 10^{-23} \text{ J/K} \times 10^7 \text{ K} \approx 10^{-16} \text{ J}$$

compare to neutron rest energy

$$mc^2 = 1.6 \times 10^{-27} \times 9 \times 10^{16} \text{ kg m/sec}^2 \approx 10^{-10} \text{ J} \gg k_B T$$

these neutrons are nonrelativistic (by a factor of 10^6 or so)

(b) compute Fermi energy for nonrelativistic gas

$$N = \frac{2 \cdot 4\pi V p_F^3}{h^3 3}$$

so

$$\epsilon_F = \frac{p_F^2}{2m} = \frac{h^2}{2m} \left(\frac{3 N}{8\pi V} \right)^{2/3}$$

plug in numbers, compute density $\rho = 10^{14} \text{ g/cm}^3 = 10^{17} \text{ kg/m}^3$

$N/V = \rho/m = (10^{17} \text{ kg/m}^3)/(1.6 \times 10^{-27} \text{ kg}) \approx 10^{44}/\text{m}^3$ so

$$\epsilon_F \approx 10^{-67} \text{ J}^2 \text{ sec}^2 \times (10^{43} \text{ m}^{-3})^{2/3} / (10^{-27} \text{ kg}) \approx 10^{-11} \text{ J} \gg k_B T$$

so neutron star can be considered a zero-temperature Fermi gas

(c) ground state energy of Fermi gas

$$E = \frac{3}{5} N \epsilon_F \approx N \epsilon_F$$

pressure is volume derivative

$$p = - \left(\frac{\partial E}{\partial V} \right) = \frac{2}{3} \frac{E}{V} = \frac{2}{5} N \epsilon_F / V$$

estimate

$$p \approx \epsilon_F N / V \approx 10^{-11} \text{ J} \times 10^{44} \text{ m}^{-3} = 10^{33} \text{ Pa}$$

(d) pressure is generated gravitationally, so if the star has radius R we can estimate

$$p \approx GM^2/R^4 = GM^{2/3}(M/R^3)^{4/3}$$

so

$$M = \left(\frac{p}{G} \right)^{3/2} \frac{1}{\rho^2}$$

plug in to find

$$M \approx (10^{33} \text{ N/m}^2 / 10^{-10} \text{ N m}^2/\text{kg}^2)^{3/2} / (10^{34} \text{ kg}^2/\text{m}^6) \approx 10^{29} \text{ kg}$$

is the mass needed to generate this pressure.

Note that this mass corresponds to a neutron star size of $R \approx 10^4$ m, and is roughly the mass of our sun (which is 2×10^{30} kg).

More careful analysis shows that the critical mass needed to create a neutron star (to force electrons to combine with protons) is about 3×10^{30} kg.

4. (a) During the free expansion, no work is done on the gas; also no heat is transferred to the gas. Then, by the first law of thermodynamics ($\Delta E = \Delta W + \Delta Q$) we conclude that the internal energy E does not change.

(b) For an ideal gas, $E = 3Nk_B T/2$, so the temperature does not change during a free expansion. Since the ideal gas entropy is

$$S(N, T, V) = Nk_B \left[\ln V + \frac{3}{2} \ln T + \text{const} \right]$$

we have an entropy change of $Nk_B \ln V_f/V_i$. The sign reflects the fact that the process is irreversible; the amount reflects the fact that the number of positional states has increased by a factor V_f/V_i per gas molecule.

(c) The free expansion is an irreversible process (the entropy change is positive) but it is subject to the constraint that E be fixed. Writing the equilibrium equation of state for the gas $T(E, V)$, we see that since E is fixed, the final value of temperature is uniquely determined, $T_f = T(E, V_f)$. Therefore we can compute T_f by considering a quasi-equilibrium and reversible path: we imagine keeping E fixed while slowly increasing V .

We therefore consider the change of temperature with volume, at fixed internal energy E .

$$\left(\frac{\partial T}{\partial V} \right)_E = - \left(\frac{\partial E}{\partial V} \right)_T \left(\frac{\partial T}{\partial E} \right)_V$$

We can determine the unknown derivatives using the first law in the form

$$dE = TdS - pdV$$

which holds even for an irreversible process. This tells us

$$\left(\frac{\partial E}{\partial V} \right)_T = T \left(\frac{\partial S}{\partial V} \right)_T - p \quad \text{and} \quad \left(\frac{\partial E}{\partial T} \right)_V = T \left(\frac{\partial S}{\partial T} \right)_V = C_V$$

Plug together to obtain

$$\left(\frac{\partial T}{\partial V} \right)_E = -\frac{1}{C_V} \left[T \left(\frac{\partial S}{\partial V} \right)_T - p \right]$$

and use the Maxwell relation $(\partial S/\partial V)_T = (\partial p/\partial T)_V$ to find

$$\left(\frac{\partial T}{\partial V} \right)_E = -\frac{1}{C_V} \left[T \left(\frac{\partial p}{\partial T} \right)_V - p \right] = -\frac{T^2}{C_V} \left[\frac{1}{T} \left(\frac{\partial p}{\partial T} \right)_V - \frac{p}{T^2} \right]$$

or finally

$$\left(\frac{\partial T}{\partial V} \right)_E = -\frac{T^2}{C_V} \left[\left(\frac{\partial p/T}{\partial T} \right)_V \right]$$

(d) Go back to

$$\left(\frac{\partial T}{\partial V}\right)_E = -\left(\frac{\partial T}{\partial E}\right)_V \left(\frac{\partial E}{\partial V}\right)_T = -\frac{1}{C_V} \left(\frac{\partial E}{\partial V}\right)_T$$

The specific heat is always positive (it is mean-square fluctuation of energy); on the other hand, when a gas is expanded at constant temperature, its internal energy will always go down, since the molecules are on average farther apart, and the repulsive interactions (keeping it in the gas phase) become weaker. Therefore we must have $\left(\frac{\partial E}{\partial V}\right)_T > 0$. The free expansion of a gas therefore always reduces its temperature.

5. (a) For one spin, $Z_1 = e^{\beta\mu B} + e^{-\beta\mu B} = 2 \cosh \beta\mu B$
 For N spins, $Z_N = (Z_1)^N = (2 \cosh \beta\mu B)^N$, so free energy is
 $-F = k_B T \ln Z_N = N k_B T [\ln \cosh \beta\mu B + \ln 2]$
 Calculate $S = -(\partial F / \partial T)_B$

$$\frac{S}{N} = k_B \left[\ln \cosh \frac{\mu B}{k_B T} + \ln 2 - \frac{\mu B}{k_B T} \tanh \frac{\mu B}{k_B T} \right]$$

(b) If $B \neq 0$, then we have $S \rightarrow 0$ as $T \rightarrow 0$ in accord with third law. But, for the case $B = 0$, we have $S \rightarrow k_B \ln 2$ as $T \rightarrow 0$, violating the third law. The origin of this is the highly degenerate ground state for $B = 0$; all states actually have the same (zero) energy in this case.

(c) When we cool the crystal in a field B_i to temperature T_i , the entropy per spin will reach

$$\frac{S(B_i, T_i)}{N} = k_B \left[\ln 2 \cosh \frac{\mu B_i}{k_B T_i} - \frac{\mu B_i}{k_B T_i} \tanh \frac{\mu B_i}{k_B T_i} \right]$$

which will be near to zero if $\mu B_i \gg k_B T_i$.

If the spins are thermally isolated and then the field is slowly reduced, the entropy of the spins must stay fixed. However, this means that when the field reaches B_2 , we must have

$$S(B_1, T_1)/N = S(B_2, T_2)/N$$

Since the entropy is a function of B/T we must have $B_i/T_i = B_f/T_f$, or $T_f/T_i = B_f/B_i$.

Thus, if $B_f < B_i$, $T_f < T_i$. This effect is often used to cool solids to very low temperature ($< 0.1K$).