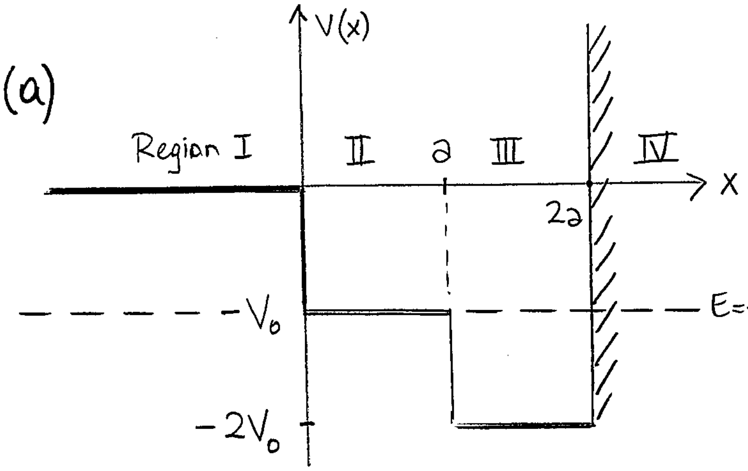
QUANTUM MECHANICS QUALIFYING EXAM (Jan '06)

SOLUTIONS

① (a)



a) Region I

$$\frac{d^2\phi_1}{dx^2} = \frac{2m}{\hbar^2}\left(V(x)-E\right)\phi_1(x) = \overbrace{\frac{2mV_0}{\hbar^2}}^{k^2}\phi_1(x)$$

$$\Rightarrow \underline{\underline{\phi_1(x) = Ae^{kx} + Be^{-kx}}}$$

Region II: $\frac{d^2\phi_2}{dx^2} = \frac{2m}{\hbar^2}\left(-V_0+V_0\right)\phi_2(x) = 0 \Rightarrow \underline{\underline{\phi_2(x) = C_1 + C_2 x}}$

Region III $\frac{d^2\phi_3}{dx^2} = \frac{2m}{\hbar^2}\left(-2V_0+V_0\right)\phi_3(x) = -k^2\phi_3(x) \Rightarrow \underline{\underline{\phi_3(x) = D'\sin kx + E'\cos kx}}$

or $\underline{\underline{\phi_3(x) = D\sin(kx+\phi)}}$

Region IV $\phi_4(x) = 0$

b) ∂-conditions:

Square-integrability of ϕ (or ∂-condition at $x=-\infty$) $\Rightarrow B=0$

Continuity of ϕ at $x=2a \Rightarrow D\sin(k2a+\phi)=0 \Rightarrow \phi = -2ka$

At $x=0 \Rightarrow$ Continuity of $\phi \Rightarrow Ae^{0} = C_1 \Rightarrow C_1 = A$ (1)

$\Rightarrow$ " of $\phi' \Rightarrow kA = C_2$ (2)

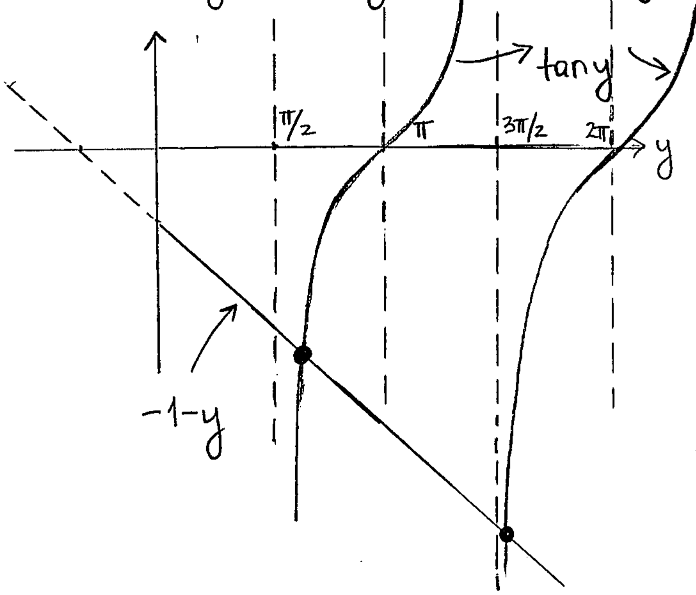
At $x=a \Rightarrow$ Continuity of $\phi \Rightarrow C_1 + C_2 a = D\sin(ka-2ka) = -D\sin ka$ (3)

" " $\phi' \Rightarrow C_2 = Dk\cos(ka-2ka) = Dk\cos ka$ (4)

$$\frac{(3)}{(4)} \Rightarrow -\frac{1}{\cancel{k}}\tan \underbrace{ka}_{y} = \frac{C_1+C_2 a}{C_2} \xrightarrow[\text{(1) \& (2)}]{\text{use}} = \frac{\cancel{A}+k\cancel{A}a}{k\cancel{A}} = \frac{1}{\cancel{k}}(1+\underbrace{ka}_{y}) \Rightarrow \underline{\underline{\tan y = -(1+y)}}$$

$$\underline{\underline{f(y) = -1-y}}$$

c.) Plot $\tan y$ & $-1-y$ on the same graph for $y>0$



The intersection points ⇒ allowed V_0's such that $E=-V_0$ is an eigenstate

$$\tan y_1 = -1-y_1$$

From the plot we look for y_1 from $\frac{\pi}{2} \sim 1.57$ to 3.14, closer to $\pi/2$

try 1.7 ⇒ $\tan 1.7 \cong -7.7$ (too far from −2.7)
1.8 ⇒ $\tan 1.8 \cong -4.3$ (" " −2.8)
1.9 ⇒ $\tan 1.9 \simeq -2.93$ quite close to −2.9!

$$\Rightarrow y_1 \approx 1.9 = ka \Rightarrow (1.9)^2 = \frac{2mV_0}{\hbar^2} a^2 \Rightarrow \underline{\underline{V_{0,min} = \frac{3.61\,\hbar^2}{2ma^2}}}$$

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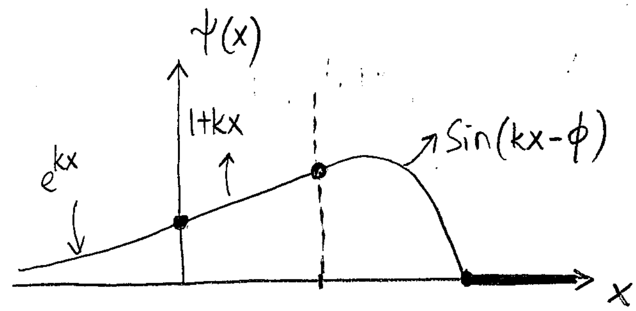
d) Ground state → no nodes

$$\psi(x) = \begin{cases} \sim e^{kx} & x<0 \\ \sim 1+kx & 0<x<a \\ \sim \sin(kx-\phi) & a<x<2a \\ 0 & x>2a \end{cases}$$



② a) In the $\{|1\rangle, |2\rangle\}$ basis, $\hat{H}$ is represented by $+\Omega \begin{pmatrix} \cos\omega t & 0 \\ 0 & -\cos\omega t \end{pmatrix}$

$|\chi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$, and $i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

$$\Rightarrow i\hbar \frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \Omega \begin{pmatrix} \cos\omega t & 0 \\ 0 & -\cos\omega t \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \Rightarrow \begin{matrix} i\hbar\dot{a}(t) = \Omega\cos\omega t\, a(t) \\ i\hbar\dot{b}(t) = -\Omega \sin\omega t\, b(t) \end{matrix}$$

$$\Rightarrow \begin{cases} \frac{da}{a} = \frac{-i\Omega\cos\omega t}{\hbar}dt \\ \frac{db}{b} = \frac{i\Omega\cos\omega t}{\hbar}dt \end{cases} \xrightarrow{\text{integrate}} \begin{matrix} \ln a(t) = \frac{-i\Omega\sin\omega t}{\hbar\omega} + \ln a(0) \\ \ln b(t) = \frac{i\Omega\sin\omega t}{\hbar\omega} + \ln b(0) \end{matrix}$$

$$\Rightarrow a(t) = a(0)\exp\left(\frac{-i\Omega}{\hbar\omega}\sin\omega t\right)$$
$$b(t) = b(0)\exp\left(\frac{i\Omega}{\hbar\omega}\sin\omega t\right)$$

all that remains is to determine $|\chi(0)\rangle$ $\Rightarrow$ diagonalize $\hat{D}$

$\hat{D}$ represented by $\begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}$ in the $\{|1\rangle, |2\rangle\}$ basis $\det(D-\lambda I)=0 \Rightarrow \begin{vmatrix} -\lambda & d \\ d & -\lambda \end{vmatrix} = 0$

$\lambda^2 = d^2 \Rightarrow \lambda = \mp d$ For $\lambda = d$ the eigenvector is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -d$ " is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ → call this $|\eta_2\rangle$ (need it for part (b))

Therefore (since $\chi(0)$ is an eigenstate of $\hat{D}$ with eigenvalue d) $a(0) = b(0) = \frac{1}{\sqrt{2}}$

$$\Rightarrow |\chi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left(\frac{-i\Omega}{\hbar\omega}\sin\omega t\right) \\ \exp\left(\frac{i\Omega}{\hbar\omega}\sin\omega t\right) \end{pmatrix}$$

b) $P(t) = |\langle \eta_2 | \chi(t)\rangle|^2$ where $\hat{D}\eta_2 = -d\eta_2$

$$= \left| \frac{1}{\sqrt{2}}(1\ -1)\frac{1}{\sqrt{2}}\begin{pmatrix}\exp(-if(t))\\ \exp(if(t))\end{pmatrix}\right|^2 = \frac{1}{4}\left|\underbrace{e^{-if(t)} - e^{if(t)}}_{-2i\sin f(t)}\right|^2 = \sin^2\left(\frac{\Omega}{\hbar\omega}\sin\omega t\right)$$

(c) $$P(t) = \sin^2\left(\frac{\Omega}{\hbar\omega}\sin\omega t\right)$$

For the measurement to yield $-d$ with 100% certainty $P(t)$ = needs to be 1

$$\Rightarrow \frac{\Omega}{\hbar\omega}\sin\omega t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots$$

But since $\sin\omega t$ is bounded by 1, for a fixed ω, Ω may not be large enough even to make $\frac{\Omega}{\hbar\omega}\sin\omega t = \pi/2$ for any t.

If we want to find the Ω_{min} set $\sin\omega t = 1$ and require $\frac{\Omega}{\hbar\omega}\cdot 1$ to be at least $\pi/2$ $\Rightarrow \frac{\Omega_{min}}{\hbar\omega}\cdot 1 = \pi/2 \Rightarrow \Omega_{min} = \frac{\hbar\omega\pi}{2}$

If this satisfied, at $t = \frac{\pi}{2\omega}, \frac{3\pi}{2\omega}, \ldots$ $P(t)$ will be 1.

③ $\vec{u} = O\vec{r} \Rightarrow \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{O} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \vec{r} = O^{-1}\vec{u} = O^T\vec{u}$

Therefore,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{O^{-1} = O^T} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \Rightarrow \begin{aligned} x &= \frac{-u_1}{\sqrt{2}} + \frac{u_2}{\sqrt{2}} \\ y &= \frac{u_1}{2} + \frac{u_2}{2} + \frac{u_3}{\sqrt{2}} \\ z &= \frac{u_1}{2} + \frac{u_2}{2} - \frac{u_3}{\sqrt{2}} \end{aligned}$$

a) $\Rightarrow \lambda x(y+z) = \frac{\lambda}{\sqrt{2}}(-u_1+u_2)(u_1+u_2) = \frac{\lambda}{\sqrt{2}}(u_2^2 - u_1^2)$

b) No need to transform $-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \frac{\hat{\vec{p}}^2}{2m}$ since O leaves $\vec{p}^2$ invariant

$\frac{1}{2}m\omega^2\vec{r}^2$ $\qquad\qquad\qquad\qquad$ $\vec{r}^2$ "

(i.e no cross terms)

Time-independent SE $\Rightarrow$

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2}\right)\psi(u_1,u_2,u_3) + \underbrace{\left[\frac{1}{2}m\omega^2(u_1^2+u_2^2+u_3^2) + \frac{\lambda}{\sqrt{2}}(u_2^2-u_1^2)\right]}_{\left(\frac{1}{2}m\omega^2 - \frac{\lambda}{\sqrt{2}}\right)u_1^2 + \left(\frac{1}{2}m\omega^2 + \frac{\lambda}{\sqrt{2}}\right)u_2^2 + \frac{1}{2}m\omega^2 u_3^2}\psi = E\psi$$

$\Rightarrow$ This is separable in u_1, u_2, u_3 and in all directions we have harmonic oscillators $\Rightarrow E_{n_1 n_2 n_3} = (n_1+\frac{1}{2})\hbar\omega_1' + (n_2+\frac{1}{2})\hbar\omega_2' + (n_3+\frac{1}{2})\hbar\omega_3'$

Set $\frac{1}{2}m\omega_1'^2 = \underbrace{\frac{1}{2}m\omega^2 - \frac{\lambda}{\sqrt{2}}}_{\text{assume } >0} \Rightarrow \omega_1'^2 = \omega^2 - \frac{\sqrt{2}\lambda}{m} \Rightarrow \omega_1' = \left(\omega^2 - \frac{\sqrt{2}\lambda}{m}\right)^{1/2}$

$$\frac{1}{2}m\omega_2'^2 = \frac{1}{2}m\omega^2 + \frac{\lambda}{\sqrt{2}} \Rightarrow \omega_2' = \left(\omega^2 + \frac{\sqrt{2}\lambda}{m}\right)^{1/2}$$

and obviously $\omega_3' = \omega \Rightarrow$

$$E_{n_1 n_2 n_3} = \left(n_1+\frac{1}{2}\right)\hbar\sqrt{\omega^2 - \frac{\sqrt{2}\lambda}{m}} + \left(n_2+\frac{1}{2}\right)\hbar\sqrt{\omega^2 + \frac{\sqrt{2}\lambda}{m}} + \left(n_3+\frac{1}{2}\right)\hbar\omega \qquad n_i = 0,1,\ldots$$

c) To have only bound states ω_1' & ω_2' have to be real, i.e

$$\frac{1}{2}m\omega^2 - \frac{\lambda}{\sqrt{2}} > 0 \Rightarrow \lambda < \frac{m\omega^2}{\sqrt{2}}$$

and

$$\frac{1}{2}m\omega^2 + \frac{\lambda}{\sqrt{2}} > 0 \Rightarrow \lambda > \frac{-m\omega^2}{\sqrt{2}}$$

$$\Rightarrow \quad \frac{-m\omega^2}{\sqrt{2}} < \lambda < \frac{m\omega^2}{\sqrt{2}}$$

d) Since $\omega_1' \neq \omega_2' \neq \omega_3'$, there is no degeneracy for a general λ (no essential degeneracy). Although $\frac{1}{2}m\omega^2\vec{r}^2$ has full rotational symmetry when $x(y+z)$ is added, the only remaining symmetries are $(xyz), (\bar{x}\bar{z}\bar{y}), (xzy), (\bar{x}\bar{y}\bar{z}) \rightarrow C_{2v}$ which has no 2 or higher dimensional irreducible representations $\Rightarrow$ no degeneracy.

If λV_1 were $\lambda(xy+yz+xz)$, then there would be three-fold rotations such as $x \to y$, $y \to z$, $z \to x$ still leaving $\hat{H}$ invariant. As a result of this symmetry, a 2D harmonic oscillator still remains $\Rightarrow$ higher symmetry $\rightarrow$ degeneracy.

4 (a) $\hat{H} = \frac{3a}{2\hbar}\hat{L}_z - \frac{a}{\hbar^2}\left(\hat{L}_x^2 + \hat{L}_y^2\right) = \frac{3a}{2\hbar}\hat{L}_z - \frac{a}{\hbar^2}\left(\hat{\vec{L}}^2 - \hat{L}_z^2\right)$.

Since $[\hat{H}, \hat{\vec{L}}^2] = [\hat{H}, \hat{L}_z] = 0$, the $|\ell m\rangle$ states are eigenstate of $\hat{H}$. With $\ell = 2$, $-2 \leq m \leq 2$

$$\Rightarrow \quad \hat{H}|2,m\rangle = \frac{3a}{2\hbar}\hat{L}_z|2,m\rangle - \frac{a}{\hbar^2}\hat{\vec{L}}^2|2,m\rangle + \frac{a}{\hbar^2}\hat{L}_z^2|2,m\rangle$$

$$= \left(\frac{3a}{2\hbar}m\hbar - \frac{a}{\hbar^2}2(2+1)\hbar^2 + \frac{a}{\hbar^2}m^2\hbar^2\right)|2,m\rangle$$

$$= \left(\frac{3a}{2}m - 6a + am^2\right)|2,m\rangle \Rightarrow \underline{\underline{E_m = a\left(\frac{3m}{2} + m^2 - 6\right) \quad m = -2, \dots, +2}}$$

$\underline{\underline{E_2 = a}}$, $\underline{\underline{E_1 = -3.5a}}$, $\underline{\underline{E_0 = -6a}}$, $\underline{\underline{E_{-1} = -6.5a}}$, $\underline{\underline{E_{-2} = -5a}}$

ground state

(b) $\psi(\theta,\phi) = A\left(\underbrace{\sin\theta\cos\theta\cos\phi}_{\text{related to } Y_{2,\mp1}} + \underbrace{\sin^2\theta\sin\phi\cos\phi}_{\text{related to } Y_{2,\mp2}}\right)$ since $\sin\phi\cos\phi = \frac{1}{2}\sin 2\phi$

- $Y_{2,+1} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi}$ $\quad Y_{2,-1} = +\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\phi}$ $\Rightarrow Y_{2,1} - Y_{2,-1} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, 2\cos\phi$

$$\Rightarrow \underline{\underline{\sin\theta\cos\theta\cos\phi}} = -\sqrt{\frac{2\pi}{15}}\left(Y_{2,1} - Y_{2,-1}\right) = \sqrt{\frac{2\pi}{15}}\left(Y_{2,-1} - Y_{2,+1}\right)$$

- $Y_{2,+2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}$ $\quad Y_{2,-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-i2\phi}$ $\Rightarrow Y_{2,2} - Y_{2,-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\left(2i\sin 2\phi\right)$

$$\Rightarrow \sin^2\theta\sin\phi\cos\phi = \frac{1}{2}\sin^2\theta\sin 2\phi = -i\sqrt{\frac{2\pi}{15}}\left(Y_{2,2} - Y_{2,-2}\right)$$

$\langle\theta,\phi|\Psi\rangle$ is $\psi(\theta,\phi)$

$$\Rightarrow |\Psi\rangle = A\sqrt{\frac{2\pi}{15}}\left(|2,-1\rangle - |2,+1\rangle + i|2,-2\rangle - i|2,2\rangle\right)$$

Since $|2,m\rangle$'s are orthonormal $\quad A^2\frac{2\pi}{15}(1+1+1+1)=1 \Rightarrow A=\sqrt{\frac{15}{8\pi}}$

$$\Rightarrow |\Psi\rangle = \frac{1}{2}\Big(|2,-1\rangle - |2,+1\rangle + i|2,-2\rangle - i|2,2\rangle\Big)$$

$$\langle\Psi|\hat{H}|\Psi\rangle = \frac{1}{4}\left(E_{-1}+E_{+1}+E_{-2}+E_{+2}\right) = \frac{1}{4}\left(-6.5a-3.5a-5a+a\right) = \underline{\underline{-3.5a}}$$

(c) Initial state is $|2,-1\rangle$

$$C_f^{(1)} = \frac{-i}{\hbar}\int_0^\infty dt\, e^{i\omega_{fi}t}\, e^{-t/T}\,\frac{\lambda}{\hbar}\langle f|\hat{L}_x|2,-1\rangle = \frac{-i\lambda}{2\hbar^2}\Big[\langle f|\hat{L}_+|2,-1\rangle + \langle f|L_-|2,-1\rangle\Big] \otimes$$

$$\int_0^\infty dt\, e^{(i\omega_{fi}-\frac{1}{T})t}$$

- Now, since $\hat{L}_+|2,-1\rangle = \hbar\sqrt{2.3-(-1)(-1+1)}\,|2,0\rangle = \sqrt{6}\hbar|2,0\rangle$
  and $\hat{L}_-|2,-1\rangle = \hbar\sqrt{2.3-(-1)(-1-1)}\,|2,-2\rangle = 2\hbar|2,-2\rangle$

only $|f\rangle=|2,0\rangle$ or $|f\rangle=|2,-2\rangle$ are allowed in 1st order

$$\text{Also}\quad \int_0^\infty dt\, e^{(i\omega_{fi}-\frac{1}{T})t} = \frac{1}{i\omega_{fi}-\frac{1}{T}}\, e^{(i\omega_{fi}-\frac{1}{T})t}\Big|_0^\infty = \frac{1}{-i\omega_{fi}+\frac{1}{T}} = \frac{T}{1-i\omega_{fi}T}$$

Notice for $|f\rangle=|2,0\rangle$ $\omega_{fi}=\frac{-6a+6.5a}{\hbar}=\frac{a}{2\hbar}$ and for $|f\rangle=|2,-2\rangle$ $\omega_{fi}=\frac{-5a+6.5a}{\hbar}=\frac{3a}{2\hbar}$

$$\Rightarrow C^{(1)}_{|2,-2\rangle} = \frac{-i\lambda}{2\hbar^2}\cdot 2\hbar\,\frac{T}{1-\frac{3aiT}{2\hbar}} \Rightarrow \underline{\underline{P_{|2,-1\rangle\to|2,-2\rangle} = \frac{\lambda^2T^2}{\hbar^2\left(1+\frac{9a^2T^2}{4\hbar^2}\right)}}}$$

and

$$\Rightarrow C^{(1)}_{|2,0\rangle} = \frac{-i\lambda}{2\hbar^2}\cdot\sqrt{6}\hbar\,\frac{T}{1-\frac{aiT}{2\hbar}} \Rightarrow \underline{\underline{P_{|2,-1\rangle\to|2,0\rangle} = \frac{3\lambda^2T^2}{2\hbar^2\left(1+\frac{a^2T^2}{4\hbar^2}\right)}}}$$

⑤ a) Eigenvalues of $\vec{S}\cdot\hat{n}$ operator are $\mp\hbar/2$. Enough to work with only one eigenvalue/eigenvector as long as states are normalized.

$$\vec{S}\cdot\hat{n} = \hat{S}_x \frac{\sqrt{3}}{2} + \frac{\hat{S}_z}{2} = \frac{\hbar}{2}\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$(\vec{S}\cdot\hat{n})\,\eta_+ = \frac{\hbar}{2}\eta_+ \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \frac{a}{2} + \frac{b\sqrt{3}}{2} = a \Rightarrow b\sqrt{3} = a$$

$$\Rightarrow \text{Normalized } \eta_+ = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$\text{Initial normalized spinor} = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3i \end{pmatrix} \leftarrow \chi$$

For those which end up in $+\hbar/2$ eigenstate of $\vec{S}\cdot\hat{n}$, the final spinor is $\eta_+ = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

$$P_+ = |\langle \eta_+ | \chi \rangle|^2 = \left| \frac{1}{2}(\sqrt{3} \;\; 1)\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3i \end{pmatrix} \right|^2 = \frac{1}{40}\,|\sqrt{3} - 3i|^2 = \frac{12}{40} = \frac{3}{10} \Rightarrow P_- = \frac{7}{10}$$

$$\text{Since } P_- > P_+ \;, \quad \left.\begin{matrix} N_1 \propto P_- \\ N_2 \propto P_+ \end{matrix}\right\} \; \underline{\underline{N_1/N_2 = 7/3}}$$

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b) $\chi_1\chi_2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}_1 \begin{pmatrix} 1 \\ i \end{pmatrix}_2$ $\Rightarrow$ In the uncoupled $|m_1, m_2\rangle$ representation, this can be written as

$$\chi_1\chi_2 = \frac{1}{2\sqrt{2}}\left\{ |\tfrac{1}{2}\,\tfrac{1}{2}\rangle + i\,|\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle + \sqrt{3}\,|{-\tfrac{1}{2}}\,\tfrac{1}{2}\rangle + \sqrt{3}\,i\,|{-\tfrac{1}{2}}, {-\tfrac{1}{2}}\rangle \right\} \quad \text{(Normalized)}$$

To find the probability of measuring $|\vec{S}_1 + \vec{S}_2|^2$ as $2\hbar^2$, we need to write this state in terms of $\vec{S}^2$ eigenstates, which are the coupled representation states. The coupled $|S, M\rangle$ states are given in terms of $|m_1, m_2\rangle$ in the back. Need inverse relations $\Rightarrow$

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = |1, 1\rangle \qquad |\tfrac{1}{2}\,{-\tfrac{1}{2}}\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\rangle + |0\,0\rangle\right)$$

$$|{-\tfrac{1}{2}}\,{-\tfrac{1}{2}}\rangle = |1, -1\rangle \qquad |{-\tfrac{1}{2}}\,\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\rangle - |0\,0\rangle\right)$$

$$\Rightarrow \chi_1\chi_2 = \frac{1}{2\sqrt{2}}\left\{ |1,1\rangle + \frac{i}{\sqrt{2}}|1,0\rangle + \frac{i}{\sqrt{2}}|0,0\rangle + \frac{\sqrt{3}}{\sqrt{2}}|1,0\rangle - \frac{\sqrt{3}}{\sqrt{2}}|0,0\rangle + \sqrt{3}\,i\,|1,-1\rangle \right\}$$

Probability of $S=1 \Rightarrow \frac{1}{8}\left(1+\frac{1}{2}+\frac{3}{2}+3\right) = \frac{6}{8} = \underline{\underline{\frac{3}{4}}}$ ✓

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c) Two particles in the energy eigenstate of $3\hbar\omega \Rightarrow$ Since $E_{n_1,n_2} = \hbar\omega(n_1+n_2+1)$

then $n_1+n_2 = 2 \Rightarrow$

n_1	n_2	
0	2	$\rightarrow$ can make symmetric and antisymmetric spatial wavefunctions
2	0	
1	1	$\rightarrow$ " " only symmetric spatial wavefunctions.

Now if $\langle \hat{S}_z\rangle = 0 \Rightarrow$ either the $|1,0\rangle$ coupled state $\rightarrow$ symmetric spin state

(S M; $\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2} -\tfrac{1}{2}\rangle + |-\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right)$ uncoupled)

or

$|0,0\rangle$ " " $\rightarrow$ antisymmetric spin state

($\frac{1}{\sqrt{2}}\left(|\tfrac{1}{2} -\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$ uncoupled)

Since these are fermions, the total wavefunction has to be anti-symmetric

If the spin part is anti-symmetric, $|00\rangle = \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle - |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$ need a symmetric spatial part $\Rightarrow$ 2 possibilities

$$\underline{\underline{\psi_1(x_1,x_2;s_1s_2) = \phi_1(x_1)\,\phi_1(x_2) \otimes \frac{1}{\sqrt{2}}\left(|\tfrac{1}{2} -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right)}}$$ or

$$\underline{\underline{\psi_2(x_1,x_2;s_1,s_2) = \frac{1}{2}\left[\phi_0(x_1)\phi_2(x_2) + \phi_2(x_1)\phi_0(x_2)\right] \otimes \left(|\tfrac{1}{2} -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right)}}$$

If the spin part is symmetric $\Rightarrow$ need an anti-symmetric spatial part $\Rightarrow$ 1 possibility

$$\underline{\underline{\psi_3(x_1,x_2;s_1s_2) = \frac{1}{2}\left[\phi_0(x_1)\,\phi_2(x_2) - \phi_0(x_2)\,\phi_2(x_1)\right] \otimes \left(|\tfrac{1}{2} -\tfrac{1}{2}\rangle + |-\tfrac{1}{2},\tfrac{1}{2}\rangle\right)}}$$

Note: They can also write this as $\frac{1}{\sqrt{2(2A^2+B^2)}}\left[A\phi_0(x_1)\phi_2(x_2) + A\phi_2(x_1)\phi_0(x_2) + B\phi_1(x_1)\phi_1(x_2)\right]\left(|\tfrac{1}{2} -\tfrac{1}{2}\rangle - |-\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right)$