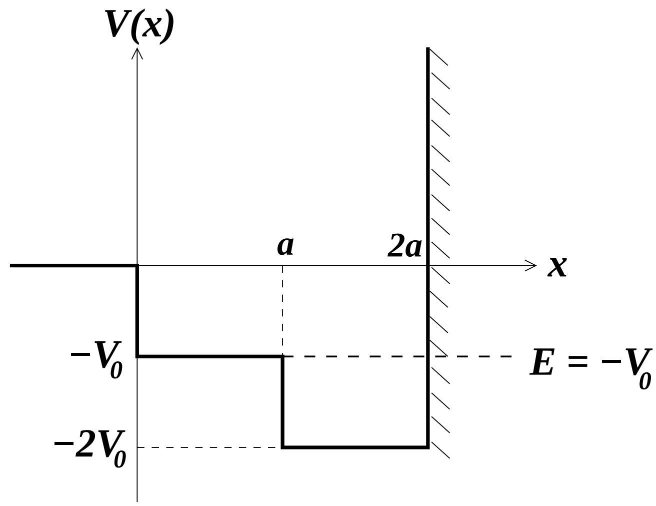
1. A particle of mass m is in the following one-dimensional potential $V(x)$, where $V_0 > 0$:

$$V(x) = \begin{cases} 0 & x < 0 \\ -V_0 & 0 < x < a \\ -2V_0 & a < x < 2a \\ \infty & x > 2a \end{cases}$$



Your main goal in this problem will be to determine the values of V_0 for which a bound eigenstate occurs at an energy $E = -V_0$.

(a) Assume that the particle is in an energy eigenstate with energy $E = -V_0$. In that case, write down the most general solutions of the Schrödinger equation in all regions of space (Use the variable k in writing down the solutions where $k^2 = 2mV_0/\hbar^2$).

(b) Apply all boundary conditions to arrive at a transcendental equation for the existence of an eigenstate at energy $E = -V_0$. With the variable y defined as $y = ka$, the transcendental equation will be of the following form where the function $f(y)$ is to be determined.

$$\tan y = f(y)$$

(c) Show how you would solve for the values of V_0 which result in an eigenstate at energy $E = -V_0$ and estimate the smallest value of V_0 in terms of m and a.

(d) For the smallest value of V_0 calculated in part (c), the energy eigenstate at $E = -V_0$ corresponds to the ground state. With this information, qualitatively sketch the corresponding wavefunction (You do not need to calculate the wavefunction explicitly).

2. Consider a time-dependent Hamiltonian for a two-state quantum system, represented in the $\{|1\rangle, |2\rangle\}$ complete and orthonormal basis set as follows:

$$\hat{H} = \Omega \cos \omega t \left(|1\rangle\langle 1| - |2\rangle\langle 2| \right)$$

Consider an operator $\hat{D}$ whose action on $\{|1\rangle, |2\rangle\}$ basis vectors is given as follows:

$$\begin{aligned} \hat{D}\,|1\rangle &= d\,|2\rangle \\ \hat{D}\,|2\rangle &= d\,|1\rangle \end{aligned}$$

where d is a real positive constant. Let us assume that the system is initially ($t = 0$) in the eigenstate of the $\hat{D}$ operator with eigenvalue d.

(a) Write down the time-dependent Schrödinger equation for the time-dependent state vector

$$|\psi(t)\rangle = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}$$

with the given Hamiltonian. Using the information about the initial ($t = 0$) state of the system given above, solve for the state vector $|\psi(t)\rangle$.

(b) At a later time t, a $\hat{D}-$measurement is carried on the system. Calculate the probability $P(t)$ of obtaining the other ($\neq d$) eigenvalue of $\hat{D}$ in this measurement.

(c) For a given ω, unless the parameter Ω is larger than a certain value Ω_{min}, the $\hat{D}-$measurement performed at any time t will not yield with 100 % certainty the other eigenvalue ($\neq d$). Calculate Ω_{min} in terms of the parameters of this problem.

3. Consider a particle moving under the following three-dimensional potential

$$V(x,y,z) = V_0(x,y,z) + \lambda V_1(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2) + \lambda x(y+z)$$

where the range of λ is such that the Hamiltonian only has bound states. Your goal in this problem is to calculate *exactly* the energy spectrum of the corresponding Hamiltonian using *Cartesian* coordinates. Obviously, the potential is not separable in the regular (x, y, z) coordinate system. However, the following *orthogonal* transformation to new coordinates (u_1, u_2, u_3) will prove useful:

$$\begin{aligned} u_1 &= -\frac{x}{\sqrt{2}} + \frac{y}{2} + \frac{z}{2} \\ u_2 &= \frac{x}{\sqrt{2}} + \frac{y}{2} + \frac{z}{2} \\ u_3 &= \frac{1}{\sqrt{2}}(y-z) \end{aligned}$$

The matrix O corresponding to the transformation to $\mathbf{u}$ from $\mathbf{r}$, that is, $\mathbf{u} = O\mathbf{r}$, is an orthogonal matrix, which has the property that $OO^T = I$, where O^T denotes the transpose of O.

(a) Express V_1 in the coordinate system u_1, u_2, u_3, and show that it results in a separable V_1.

(b) Using new $V_1(u_1, u_2, u_3)$, rewrite the *full* time-independent Schrödinger equation in the Cartesian $\mathbf{u}$−representation as a second order partial differential equation to be solved for $\psi(u_1, u_2, u_3)$ (**Hint:** An orthogonal transformation leaves the length of a vector such as $\mathbf{r}$ or $\mathbf{p}$ invariant). Without solving the equation, recognize its form, and write down the energy eigenvalues of the Hamiltonian.

(c) What is the range on λ such that the Hamiltonian has only bound states?

(d) For a general value of λ, is there any degeneracy in the spectrum? If there were an additional yz term in V_1, that is, if λV_1 were $\lambda(xy + xz + yz)$, would there be degeneracy? Explain briefly.

4. Consider the following Hamiltonian for a spinless particle with orbital angular momentum $l = 2$.

$$\hat{H} = \frac{3a}{2\hbar}\hat{L}_z - \frac{a}{\hbar^2}(\hat{L}_x^2 + \hat{L}_y^2)$$

where $a > 0$ is a constant, and $\hat{L}_i$ denotes the i^{th} component of the angular momentum operator.

(a) Calculate the energy spectrum of this Hamiltonian.

(b) Suppose a particle with this Hamiltonian has the wavefunction

$$\psi(\theta, \phi) = A \sin\theta \sin\phi \Big(\cos\theta + \sin\theta\cos\phi \Big)$$

where θ is the polar angle, ϕ is the azimuthal angle, and A is a normalization constant. What is the average energy obtained in energy measurements on an ensemble of particles described by the same $\psi(\theta, \phi)$ above?

(c) Let us now assume the particle is in the lowest energy state (with $l = 2$) for $t < 0$. Starting at $t = 0$, an external magnetic field varying exponentially with a decay constant τ is applied in the $xy-$plane along the x direction so that the time-dependent perturbation can be written for $t > 0$ as

$$\hat{V}(t) = \frac{\lambda}{\hbar}\hat{L}_x\, e^{-t/\tau}$$

Calculate the transition probabilities to possible excited states after a very long time ($\tau \ll t \to \infty$) using first order time-dependent perturbation theory.

5. The three parts **(a)**, **(b)**, and **(c)** of this problem are independent of each other.

(a) A large number of electrons, all of which are described the spinor $\chi = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3i \end{pmatrix}$ in the S_z-representation pass through a Stern-Gerlach apparatus, in which the magnetic field points along the unit vector $\hat{\mathbf{n}} = (\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ direction, and split into two beams with electron numbers N_1 and N_2. If $N_1 > N_2$ calculate the ratio N_1/N_2.

(b) Consider a system composed of a proton and a neutron. In the S_z representation, the spinors of the two particles are known to be $\chi_p = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ and $\chi_n = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, respectively.

If a measurement of $\mathbf{S}^2$ is made, where $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ is the total spin of the composite system, what is the probability that a value of $2\hbar^2$ will be obtained?

(c) Now suppose that you have two *identical* non-interacting spin-1/2 particles in an external one-dimensional harmonic oscillator potential of angular frequency ω. Let $\phi_n(x_i)$ denote the position wavefunction for the normalized n^{th} eigenstate of the i^{th} particle under a harmonic oscillator potential. The two-particle system is in an energy eigenstate with eigenvalue $3\hbar\omega$. If the $z-$component of the total spin of the particles (S_z) is zero, write down all possible normalized total wavefunctions (as a product of spatial and spin parts) for the system.

Some Possibly Useful Information

- $Y_{0,0} = \frac{1}{\sqrt{4\pi}} \quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \quad Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos\theta$

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta e^{2i\phi} \quad Y_{2,1} = -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta e^{i\phi} \quad Y_{2,0} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$

- $Y_{l,-m} = (-1)^m Y^*_{lm}$

- $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$

- $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $\hat{L}^2|lm\rangle = l(l+1)\hbar^2 \; |lm\rangle$
- $\hat{L}_z|lm\rangle = m\hbar \; |lm\rangle$
- $\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y$
- $\hat{L}_+|lm\rangle = \hbar\sqrt{l(l+1) - m(m+1)} \; |l, m+1\rangle$
- $\hat{L}_-|lm\rangle = \hbar\sqrt{l(l+1) - m(m-1)} \; |l, m-1\rangle$

- $c_f^{(1)} = -\frac{i}{\hbar}\int_0^t dt' e^{i\omega_{fi}t'} \langle f|\hat{V}(t')|i\rangle$

- For $s_1 = s_2 = 1/2$, $|S, M\rangle$ in the coupled representation are given in terms of the uncoupled basis vectors $|m_1, m_2\rangle$ as:

$$\begin{aligned}
|1,1\rangle &= \left|\frac{1}{2}, \frac{1}{2}\right\rangle \\
|1,-1\rangle &= \left|-\frac{1}{2}, -\frac{1}{2}\right\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right) \\
|0,0\rangle &= \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \left|-\frac{1}{2}, \frac{1}{2}\right\rangle\right)
\end{aligned}$$