University of Illinois at Chicago
Department of Physics

Electricity & Magnetism
Qualifying Examination

January 3, 2006
9.00 am – 12:00 pm

Full credit can be achieved from completely correct answers to 4 questions. If the student attempts all 5 questions, all of the answers will be graded, and the top 4 scores will be counted toward the exam's total score.

Miscellaneous Equations:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \cdots$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \cdots$$

$$\sin(x) = x - \tfrac{1}{3!}x^3 + \cdots$$

$$\cos(x) = 1 - \tfrac{1}{2}x^2 + \cdots$$

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

(1) $D_1^{\perp} - D_2^{\perp} = \sigma_f$

(2) $B_1^{\perp} - B_2^{\perp} = 0$

(3) $\vec{E}_1^{//} - \vec{E}_2^{//} = 0$

(4) $\vec{H}_1^{//} - \vec{H}_2^{//} = \vec{K}_f \times \hat{n}$ ($\hat{n}$: $2 \to 1$)

$$\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \vec{B} = \tfrac{1}{\omega}\vec{k} \times \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\varepsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C^2/Nm^2}$$

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

$$\sigma_b = \vec{P}\cdot\hat{n}, \quad \rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M}, \quad \vec{K}_b = \vec{M} \times \hat{n}$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}}{r^3}$$

$$\vec{E} = -\vec{\nabla}V - \tfrac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$V(\vec{r},t) = \int \frac{\rho(\vec{r}\,',t_r)}{4\pi\varepsilon_0 \mathscr{r}}\, d\tau'$$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t_r)}{\mathscr{r}}\, d\tau'$$

$$\mathcal{E} = \oint \vec{f}\cdot d\vec{\ell} = \oint (\vec{E} + \vec{v} \times \vec{B})\cdot d\vec{\ell}$$

$$\mathcal{E} = -\tfrac{d\Phi}{dt}, \quad \mathcal{E} = -L\tfrac{dI}{dt}$$

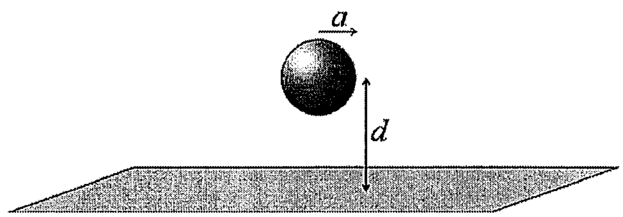
$$\vec{E} = -\vec{\nabla}V - \tfrac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

$$C \equiv \frac{Q}{V}, \quad U = \tfrac{1}{2}CV^2, \quad L \equiv \Phi_B / I, \quad U = \tfrac{1}{2}LI^2$$

$$u_{em} = \tfrac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})$$

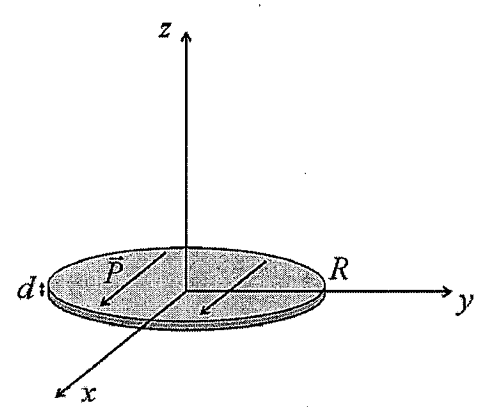
$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

1. A uniformly charged, solid, *non-conducting* sphere of radius a (charge density ρ_0) has its center located a distance d from a uniformly charged, *non-conducting*, sheet (charge density σ_0) as shown in the figure below.



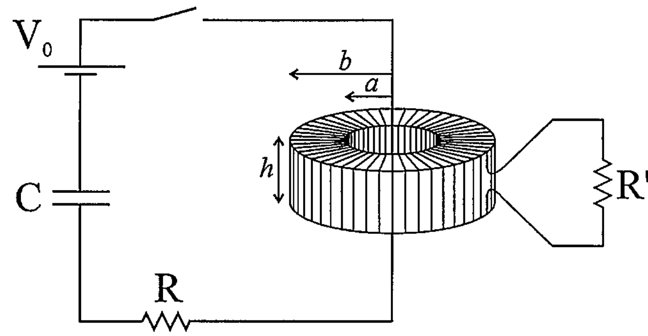
(a) Determine the potential difference between the center of the sphere and the nearest point on the charged sheet.
(b) Determine the net electrostatic force on the charged sphere.

2. A thin disc of radius R and thickness d has a uniform, frozen-in polarization along the +x axis. It lies in the xy-plane and is centered at the origin as shown in the figure below. In your calculations below, assume that $d \ll R$.



a) Compute the electric field for a point along the z-axis, **E**(z). (Do <u>not</u> assume for this part that z is large)
b) Write the electric field at the origin in terms of **P**. Explain the microscopic interpretation of this result. For example, is this an accurate description of the field near an atomic nucleus at the center of the disc? Explain why or why not?
c) Write an *approximate* expression for the electric field on the z-axis for $z \gg R$. Show that this expression has the same leading order dependence on z as your result from part (a).

3. Consider the circuits shown below. Circuit 1 consists of a resistor (R) and capacitor (C) in series with a voltage source (V_0). A long, straight wire in circuit 1 goes, along the axis, through an N-turn, toroidal coil of rectangular cross-section (inner radius = a, outer radius = b, height = h). Circuit 2 consists of the coil, having negligible resistance, in series with another resistor (R'). The capacitor is initially uncharged. At t = 0 the switch is closed.

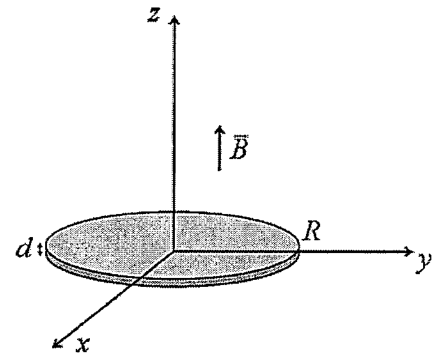


(a) What is the current $I_1(t)$ that flows through circuit 1 as a function of time? Assume that this result does not depend on any interaction with circuit 2. [It is acceptable either to just write the result, or to derive it from scratch.]

(b) What is the EMF induced in the coil as a function of time?

(c) In this situation, the current in circuit 2 is <u>not</u> accurately given by the EMF divided by R'. Why is this? Discuss and make qualitative graphs to indicate how the actual current vs. time will differ from EMF/R'.

4. Consider a thin disc of radius R and thickness d, having linear, magnetic susceptibility χ. It lies in the xy-plane and is centered at the origin as shown in the figure below. In your calculations below, assume that $d \ll R$. The disc is placed in an otherwise uniform, oscillatory magnetic field: $B_{ext}(t) = B_0\cos(\omega t)$ directed along the z-axis. Assume the susceptibility is sufficiently small that the magnetization is approximately *uniform*, with magnitude: $M = (\chi/\mu_0)B_{ext}$



(a) For this part, assume that the frequency is low, so that the field due to the disc can be calculated in the magnetostatic limit (i.e. no appreciable retardation effects). What are the constraints on ω for this magnetostatic limit? Calculate the magnetic field due to the disc at a point along the z axis.

(b) No longer assuming that the frequency is low, write an integral expression for the vector potential **A**(x,y,z,t) due to the disc, defining any symbols you use (it is not necessary to do the integration, but do work it into the form of a definite integral such that the result will be a function of x,y,z,t).

5. Consider an electromagnetic, sinusoidal plane wave, normally incident on a layer of dilute plasma from vacuum. Assume that the plasma fills the region z>0. The dispersion relation for the plasma is approximately:

$$\omega = \sqrt{\omega_p^2 + c^2 k^2}$$

where ω_p is a constant called the "plasma frequency" and c is the speed of light. You can assume the permeability of the plasma is the same as vacuum.

(a) Describe the difference in the behavior of the wave in the frequency ranges $\omega < \omega_p$ and $\omega > \omega_p$. Draw a sketch of the wave in each case, indicating the the proper relationship between wavelengths on either side of the boundary.

(b) Apply the boundary conditions (as with a normal dielectric, you can assume there is no free charge or free current anywhere), and solve for the amplitudes of the reflected and transmitted waves in terms of the incident electric field amplitude.

(c) Evaluate the amplitude of the <u>magnetic</u> field of the <u>transmitted</u> wave for the frequency:

$$\omega = \frac{\sqrt{3}}{2}\omega_p$$

Your answer should be a complex number, written in polar form, times the incident amplitude of the electric field.