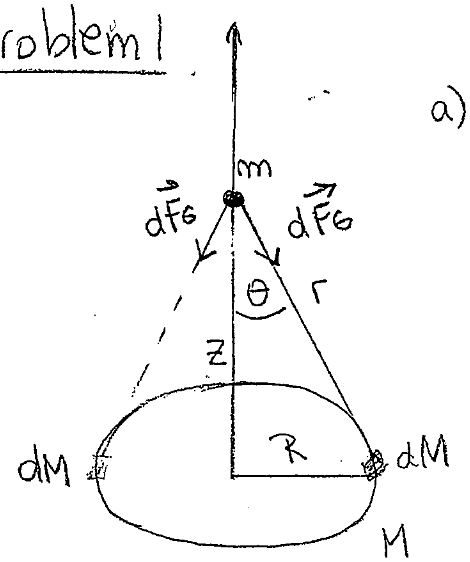
Problem 1



a) The mass M is uniformly distributed along the circumference of the ring with a linear mass density $\lambda = \frac{M}{2\pi R}$

The force exerted on the particle of mass m by dM is $d\vec{F}_G = -G\frac{m\,dM}{r^2}$

with $dM = \lambda\, dl$

But the corresponding dM on the opposite side of the ring also produces a force $d\vec{F}_G$ of the same magnitude, so that the components perpendicular to the axis cancel, but the components parallel to the axis add. Hence only the z component of $d\vec{F}_G$ survives

$$d\vec{F}_{G_z} = -\frac{G\,m\,dM\cos\theta}{r^2}\,\hat{k}$$

But $r^2 = z^2 + R^2$ $\quad \cos\theta = \frac{z}{r} = \frac{z}{\sqrt{z^2+R^2}}$

and integrating around the ring

$$F_{G_z} = -\frac{G m z}{(z^2+R^2)^{3/2}} \underbrace{\int_{ring} \lambda\, dl}_{M} \Rightarrow \boxed{\vec{F}_{G_z} = -\frac{G m M z}{(z^2+R^2)^{3/2}}\,\hat{k}}$$

b) $V(z) = -\int_{\infty}^{z} F_{Gz}\, dz = GmM \int_{\infty}^{z} \frac{z}{(z^2+R^2)^{3/2}} =$

$$= -GmM \left. \frac{1}{(z^2+R^2)^{1/2}} \right|_{\infty}^{z} = \boxed{\frac{-GmM}{(z^2+R^2)^{1/2}}}$$

c) For max F_{Gz}, $\frac{dF_{Gz}}{dz} = 0$

$$\frac{dF_G}{dz} = -\frac{GmM}{(z^2+R^2)^{3/2}} - \frac{GmMz\left(-\frac{3}{2}\right) 2z}{(z^2+R^2)^{5/2}} =$$

$$= -\frac{GmM}{(z^2+R^2)^{5/2}} \left[z^2 + R^2 - 3z^2 \right] = 0 \Rightarrow R^2 = 2z^2$$

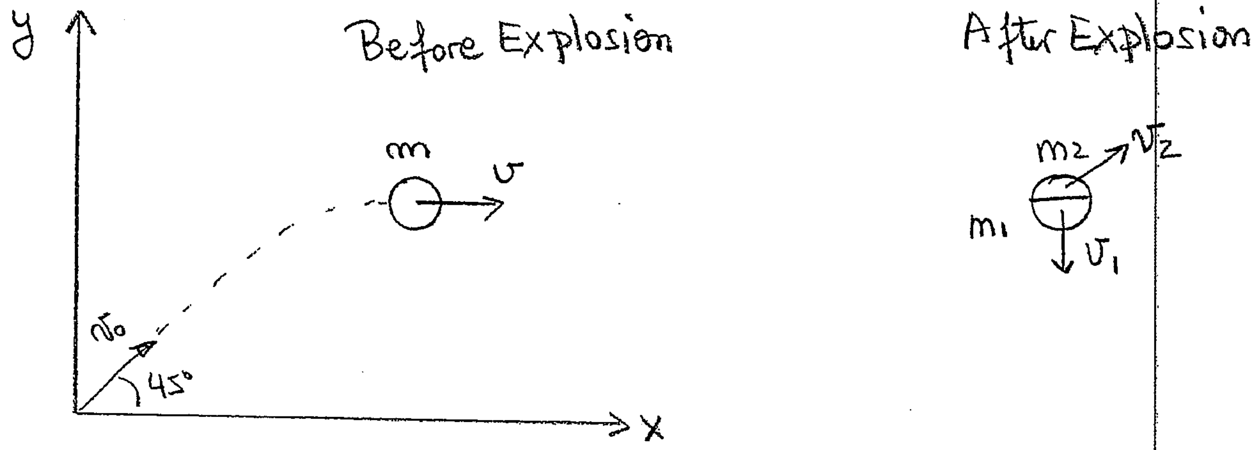
$$\Rightarrow \boxed{z = R/\sqrt{2}}$$

$$F_{Gz}^{max} = -\frac{GmM\, R/\sqrt{2}}{\left(\frac{3}{2}R^2\right)^{3/2}} = \boxed{-\frac{2}{3\sqrt{3}} \frac{GmM}{R^2}}$$

d) $z \ll R$ $\quad F_{Gz} = -\frac{GmMz}{R^3} = m\ddot{z} \Rightarrow \ddot{z} + \frac{GM}{R^3} z = 0$

Eq. for SHO with $\omega^2 = \frac{GM}{R^3}$

Problem 2



$K_0 = E_0 = \frac{1}{2} m v_0^2 \Rightarrow v_0 = \left(\frac{2E_0}{m}\right)^{1/2}$ with $m = m_1 + m_2$

Before explosion

$$v_x = v_0 \cos 45° = \frac{1}{\sqrt{2}} \sqrt{\frac{2E_0}{m}} = \sqrt{\frac{E_0}{m}}$$

$$v_y = 0$$

After explosion

$v_{1x} = 0 \quad v_{2x} \neq 0 \quad v_{1y} = -v_1 \quad v_{2y} \neq 0$

a) Conservation of linear momentum

x) $(m_1 + m_2) v_x = m_2 v_{2x}$ (1)

y) $0 = -m_1 v_1 + m_2 v_{2y}$ (2)

Conservation of energy

$$\frac{1}{2}(m_1+m_2) v_x^2 + E_0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \quad (3)$$

From (1) $v_{2x} = \dfrac{m_1+m_2}{m_2}\sqrt{\dfrac{E_0}{(m_1+m_2)}}$

$$\frac{1}{m_2}\sqrt{E_0(m_1+m_2)} = v_{2x} \quad (4)$$

From (2) $v_1 = \dfrac{m_2}{m_1} v_{2y} \Rightarrow v_{2y} = \dfrac{m_1}{m_2} v_1 \quad (5)$

From (3) $3E_0 = m_1 v_1^2 + m_2 v_2^2$

using (4) & (5) $3E_0 = m_1 v_1^2 + m_2 \left(v_{2x}^2 + v_{2y}^2 \right)$

$$3E_0 = m_1 v_1^2 + m_2 \left[\frac{E_0}{m_2^2}(m_1+m_2) + \frac{m_1^2}{m_2^2} v_1^2 \right]$$

$$3E_0 m_2 = m_1 m_2 v_1^2 + E_0(m_1+m_2) + m_1^2 v_1^2$$

$$E_0(2m_2 - m_1) = m_1 v_1^2 (m_1+m_2)$$

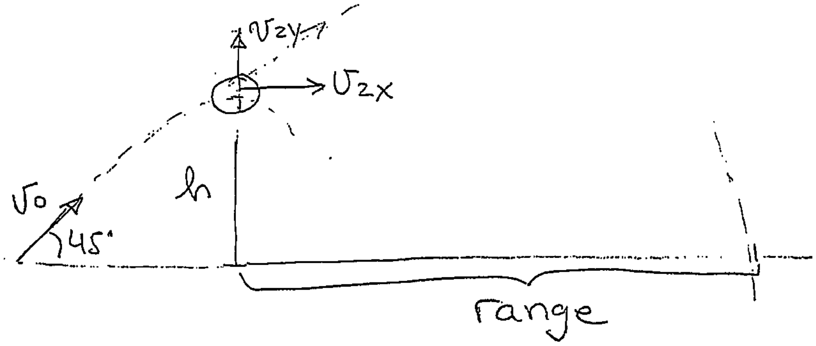
$$\boxed{v_1 = \left[\frac{E_0(2m_2 - m_1)}{m_1(m_1+m_2)} \right]^{1/2}}$$

$\therefore \boxed{v_{2y} = \left[\dfrac{E_0 m_1}{m_2^2} \dfrac{(2m_2-m_1)}{(m_1+m_2)} \right]^{1/2}}$ $\quad \boxed{v_{2x} = \dfrac{1}{m_2}\left[E_0(m_1+m_2) \right]^{1/2}}$

b) m_1 will be maximum when $v_1 = 0$.

$$\Rightarrow 2m_2 - m_1 = 0 \Rightarrow \boxed{\frac{m_1}{m_2} = 2}$$

c)



First find h from energy conservation for $m = m_1 + m_2$

$$\frac{1}{2} m v_{0y}^2 = mgh \Rightarrow h = \frac{v_{0y}^2}{2g} = \frac{E_0}{2gm}$$

Then find range for projectile motion of m_2 with initial velocity (v_{2x}, v_{2y}) and initial height h

$$x = v_{2x} t$$

$$y = h + v_{2y} t - \frac{1}{2} g t^2 = 0 \quad \text{to find horisontal range}$$

$$\Rightarrow \frac{1}{2} g t^2 - v_{2y} t - h = 0$$

$$t = \frac{v_{2y} \pm \sqrt{v_{2y}^2 + 4 \frac{1}{2} g h}}{g} \quad \text{take positive solution}$$

Range (measured from position of explosion) =

$$= v_{2x} \left(\frac{v_{2y} + \sqrt{v_{2y}^2 + 2gh}}{g} \right)$$

For $m_1 = 2\,kg$, $m_2 = 3\,kg$, $E_0 = 100\,J$, $g = 10\,m/s^2$

$v_{2x} = 7.45\,m/s \quad v_{2y} = 4.22\,m/s \quad h = 1\,m$

$$\boxed{\text{Range} = 7.72\,m}$$

Range (measured from position of launch)

$= 7.72\,m + \frac{1}{2}$ (Range of original projectile motion for $m = m_1 + m_2$) $= 7.72\,m + \frac{1}{2} R_i$

$$R_i = \frac{v_0^2 \sin 2(45^\circ)}{g} = \frac{2E_0}{(m_1+m_2)g} = 4\,m$$

$\Rightarrow$ Range (from position of launch) $= 7.72\,m + 2\,m =$

$= \boxed{9.72\,m}$

Problem 3

The ball rolls without slipping ⇒ the lengths of the arcs $PC = P'C$ ⇒

$$s\phi = a(\phi + \theta) \qquad \text{or}$$

$$(s-a)\,\phi = a\,\theta \qquad (1)$$

is the equation of constraint connecting the coordinates θ and ϕ.

The velocity of the center of the ball is $(s-a)\dot{\phi}$ and the Kinetic Energy can be written as

$$K = \frac{1}{2} M (s-a)^2 \dot{\phi}^2 + \frac{1}{2} I \dot{\theta}^2$$

but $\dot{\theta} = \frac{(s-a)}{a}\dot{\phi}$ from (1) and $I = \frac{2}{5} M a^2$

$$\Rightarrow \boxed{K = \frac{1}{2} M (s-a)^2 \frac{7}{5} \dot{\phi}^2}$$

The Potential Energy of the ball is

$$\boxed{V = Mg(s-a)(1-\cos\phi)}$$

The equation of motion for ϕ can be found using Lagrange's Equations:

$$\mathcal{L} = K - V = \frac{1}{2} M (s-a)^2 \frac{7}{5} \dot{\phi}^2 - Mg(s-a)(1-\cos\phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -Mg(s-a)\sin\phi$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = M(s-a)^2 \frac{7}{5} \ddot{\phi}$$

$\Rightarrow$ $\dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\phi}} - \dfrac{\partial \mathcal{L}}{\partial \phi} = 0$ gives

$$\frac{7}{5} \cancel{M} (s-a)^{\cancel{2}} \ddot{\phi} + \cancel{M} g \cancel{(s-a)} \sin\phi = 0$$

$$\boxed{\frac{7}{5}(s-a)\ddot{\phi} + g\sin\phi = 0}$$ Eq. of Motion for ϕ.

b) Small oscillations $\Rightarrow$ $\sin\phi \approx \phi$

$$\ddot{\phi} + \frac{5}{7}\frac{g}{s-a}\phi = 0$$ SHO with

$$\boxed{\omega = \sqrt{\frac{5}{7}\frac{g}{s-a}}}$$

c) $a \to 0$, $\omega \to \sqrt{\frac{5}{7}\frac{g}{s}}$

Differs from the frequency for a plane pendulum by the factor of $\sqrt{\frac{5}{7}}$. The constraint of rolling is responsible for this factor.

Problem 4

a) $K = \frac{1}{2}m\left[\dot{r}^2 + \dot{z}^2 + (r\dot{\theta})^2\right]$ using the expression for velocity in cylindrical coordinates.

b) $\mathcal{U} = mgz$

c) Eq of constraint for motion on parabola is

$$z = cr^2$$

Eq of constraint for rotation of wire is

$$\theta = \omega t$$

The system has $3-2=1$ degrees of freedom

d) $\mathcal{L} = K - \mathcal{U}$ Need to express K and $\mathcal{U}$ as a function of $r, \dot{r}, \omega$

$$z = cr^2 \Rightarrow \dot{z} = 2cr\dot{r}$$
$$\theta = \omega t \Rightarrow \dot{\theta} = \omega$$

$$K = \frac{1}{2}m\left[\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right]$$

$$\mathcal{U} = mgcr^2$$

$$\mathcal{L} = \frac{1}{2}m\left[\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2\right] - mgcr^2$$

Lagrange's equation of motion:

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$\frac{\partial L}{\partial r} = m\left[4c^2 r \dot{r}^2 + r\omega^2 - 2gcr\right]$$

$$\frac{\partial L}{\partial \dot{r}} = m\left[\dot{r} + 4c^2 r^2 \dot{r}\right]$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\left[\ddot{r} + 4c^2 r^2 \ddot{r} + 8c^2 r \dot{r}^2\right]$$

$$\Rightarrow 4c^2 r \dot{r}^2 + r\omega^2 - 2gcr - \ddot{r} - 4c^2 r^2 \ddot{r} - 8c^2 r \dot{r}^2 = 0$$

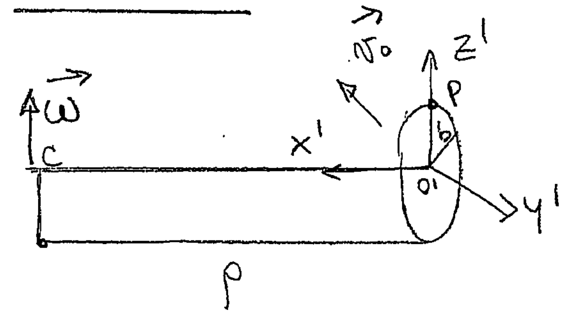
$$\boxed{\ddot{r}(1 + 4c^2 r^2) + \dot{r}^2(4c^2 r) + r(2gc - \omega^2) = 0.}$$

e) If $r = R$, $\dot{r} = \ddot{r} = 0$ Eq. reduces to

$$R(2gc - \omega^2) = 0$$

$$\Rightarrow \boxed{c = \frac{\omega^2}{2g}}$$

Problem 5



x' points towards the center of curvature C of the track

o' rotates about C

z' always vertical

$$\Rightarrow \bar{\omega} = \omega \hat{k}' \qquad \vec{v}_0 = v_0(-\hat{j})$$

a) Motion of o' w.r.t. C

o' rotates with $\vec{\omega}$ about C (circle of radius ρ)

$$\vec{\omega}_{o'c} = \vec{\omega} = \frac{v_0}{\rho}\hat{k}' \quad \text{and} \quad \vec{a}_{o'c} = \frac{v_0^2}{\rho}\hat{i}' \quad \text{from circular motion}$$

$\vec{\omega}$ is the angular velocity of o', a_c is the acceleration of o'.

b) Motion of a point at the top of the wheel w.r.t. o'

$$\vec{v}_{Po'} = -v_0\hat{j}' \qquad \vec{a}_{Po'} = -\frac{v_0^2}{b}\hat{k}'$$

c) $$\vec{a}_{\text{Coriolis}} = 2\bar{\omega}_{o'c} \times \vec{v}_{Po'} = 2\frac{v_0}{\rho}\hat{k}' \times (-v_0\hat{j}') = 2\frac{v_0^2}{\rho}\hat{i}'$$

$$\vec{a}_{\text{centripetal}} = \bar{\omega}_{o'c} \times (\bar{\omega}_{o'c} \times \vec{r}') = \frac{v_0}{\rho}\hat{k}' \times \left(\frac{v_0}{\rho}\hat{k}' \times b\hat{k}'\right) = 0$$

$$\vec{a}_{\text{transverse}} = \dot{\vec{\omega}}_{o'c} \times \vec{r}' = 0 \quad (\vec{\omega}_{o'c} = \text{constant})$$

d) $\vec{a}_{net} = \vec{a}_{PC} = \vec{a}_{PO'} + \vec{a}_{trans} + \vec{a}_{coriolis} +$

$$+ \vec{a}_{centripetal} + \vec{a}_{O'C} =$$

$$= -\frac{v_0^2}{b}\hat{k}' + 0 + \frac{2v_0^2}{\rho}\hat{\imath}' + 0 + \frac{v_0^2}{\rho}\hat{\imath}'$$

$$\boxed{\vec{a}_{PC} = \frac{3v_0^2}{\rho}\hat{\imath}' - \frac{v_0^2}{b}\hat{k}'}$$