

1. Ideal gas of two-state atoms

Set temperature units to make $k = 1$.

a) The fraction of atoms in excited state after thermalization:

$$p = \frac{1}{e^{\Delta/T_2} + 1} \quad (1)$$

Energy conservation:

$$\frac{3}{2}T_1 = \frac{3}{2}T_2 + \frac{\Delta}{e^{\Delta/T_2} + 1} \quad (2)$$

Expanding to order Δ :

$$T_2 - T_1 = -\frac{\Delta}{3} < 0 \quad (3)$$

Temperature decreases.

b) The entropy of the two-state system:

$$S_\Delta = \frac{\partial}{\partial T} (NT \ln(1 + e^{-\Delta/T})) = N \ln(1 + e^{-\Delta/T}) + N \frac{\Delta/T}{e^{\Delta/T} + 1} = N \ln 2 + \mathcal{O}\left(\frac{\Delta^2}{T^2}\right) \quad (4)$$

Alternatively, using p from (1):

$$S_\Delta/N = -p \ln p - (1-p) \ln(1-p) = \ln 2 + \mathcal{O}\left(\frac{\Delta^2}{T^2}\right), \quad (5)$$

The total entropy of the gas

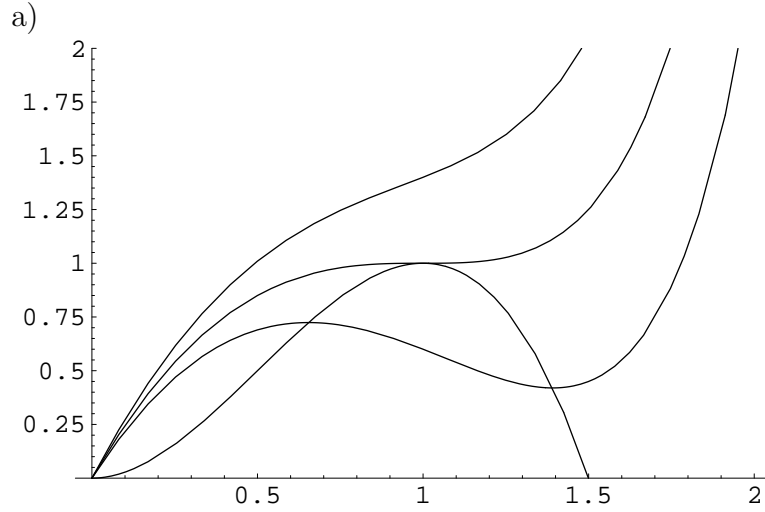
$$S = S_{\text{kin}} + S_\Delta = \frac{3}{2}N \ln T + N \ln 2 + \mathcal{O}\left(\frac{\Delta^2}{T^2}\right) \quad (6)$$

The change in the entropy to order Δ/T :

$$S_2 - S_1 = N \left(\frac{3}{2} \ln T_2 + \ln 2 - \frac{3}{2} \ln T_1 \right) = N \left(\frac{3}{2} \ln \left(1 - \frac{\Delta}{3T_1} \right) + \ln 2 \right) = N \left(\ln 2 - \frac{\Delta}{2T_1} \right) > 0 \quad (7)$$

Entropy increases.

2. van der Waals



b) Set $k = 1$. From

$$\left(\frac{\partial p}{\partial n}\right)_T = \frac{T}{(1 - bn)^2} - 2an = 0 \quad (8)$$

we find

$$T = 2an(1 - bn)^2 \quad (9)$$

and substituting (9) into the equation of state $p(n, T)$ we find

$$p = an^2 - 2abn^3. \quad (10)$$

See Figure.

c) Stability condition: $(\partial p / \partial n)_T > 0$.

d) The critical point is at the maximum of the curve given by (8). Two conditions defining the critical point: $(\partial p / \partial n)_T = 0$ and $(\partial^2 p / \partial n^2)_T = 0$. The second condition gives:

$$\left(\frac{\partial^2 p}{\partial n^2}\right)_T = \frac{2bT}{(1 - bn)^3} - 2a = 0. \quad (11)$$

Substituting into (11) the value of T from (9) and solving for n we find:

$$n_c = \frac{1}{3b}. \quad (12)$$

Substituting this into (10) and into (9) we find, respectively:

$$p_c = \frac{a}{27b^2}; \quad T_c = \frac{8a}{27b} \quad (13)$$

3. Maxwell relations

a) $dU = f dL + T dS$

b) Using the relation for the free energy: $d(U - TS) = f dL - S dT$, we find the Maxwell relation:

$$\left(\frac{\partial S}{\partial L}\right)_T = -\left(\frac{\partial f}{\partial T}\right)_L \quad (14)$$

Thus

$$\Delta Q = T \Delta S = -T \Delta L \left(\frac{\partial f}{\partial T}\right)_L = -T \Delta L \frac{\Delta f}{\Delta T} = -300 \text{ K} \cdot 2 \cdot 10^{-2} \text{ m} \frac{1.2 \text{ N}}{3 \text{ K}} = -2.4 \text{ J}. \quad (15)$$

Heat is released since $\Delta Q < 0$.

4. Cosmic microwave background

a)

$$n = 2 \int \frac{d^3 \mathbf{p} V}{(2\pi \hbar)^3} \frac{1}{e^{|\mathbf{p}|/kT} - 1} = \frac{2\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 = 4.0 \cdot 10^8 \text{ m}^{-3}; \quad N = nV = 400. \quad (16)$$

b) The density of photons with velocities \mathbf{v} inside an element of solid angle $d\Omega_{\mathbf{v}}$ is

$$dn = n \frac{d\Omega_{\mathbf{v}}}{4\pi}. \quad (17)$$

The flux of those photons is $d\mathbf{J} = dn\mathbf{v}$. The rate at which the photons with the given velocity are hitting a small surface element of area $\boldsymbol{\sigma}$ is

$$\boldsymbol{\sigma} \cdot d\mathbf{J} = \boldsymbol{\sigma} \cdot \mathbf{v} dn \quad (18)$$

The total rate from the photons with all values of \mathbf{v} is (integrate over semisphere, use $|\mathbf{v}| = c$):

$$\int \boldsymbol{\sigma} \cdot \mathbf{v} n \frac{d\Omega_{\mathbf{v}}}{4\pi} = \sigma cn \frac{2\pi}{4\pi} \int_0^1 \cos \theta d(\cos \theta) = \frac{1}{4} \sigma cn \quad (19)$$

Therefore rate per unit area is $cn/4$. The total rate for the area of the ball is

$$\Gamma = 4\pi R^2 \times \frac{cn}{4} = \pi R^2 c \frac{2\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 = 3.76 \times 10^{13} \text{ s}^{-1} \quad (20)$$

Another derivation: For the photons with velocity \mathbf{v} the ball presents a cross section equal to πR^2 . The rate at which these photons strike the ball is $\pi R^2 c dn$. Integrating over all \mathbf{v} we find $\Gamma = \pi R^2 cn$, as before.

5. Ultrarelativistic Fermi gas at $T = 0$

a) $\epsilon_F = \mu$ and $p_F = \epsilon_F/c = \mu/c$.

b)

$$n = 2 \int_{|\mathbf{p}| < p_F} \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} = \frac{1}{3\pi^2} \left(\frac{\mu}{\hbar c} \right)^3. \quad (21)$$

c)

$$E = V \cdot 2 \int_{|\mathbf{p}| < p_F} \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} |\mathbf{p}|c = V \cdot \frac{1}{4\pi^2} \frac{\mu^4}{(\hbar c)^3} = V \cdot \frac{\hbar c}{4\pi^2} (3\pi^2 n)^{4/3} = \frac{\hbar c}{4\pi^2} (3\pi^2 N)^{4/3} V^{-1/3} \quad (22)$$

where we used Eq. (21) and $n = N/V$.

d)

$$P = - \left(\frac{\partial E}{\partial V} \right)_N = \frac{\hbar c}{12\pi^2} \left(3\pi^2 \frac{N}{V} \right)^{4/3} = \frac{1}{12\pi^2} \frac{\mu^4}{(\hbar c)^3} \quad (23)$$

e) The condition for the reaction to be energetically allowed is

$$m_A c^2 + \mu > m_B c^2 \quad (24)$$

Thus $\mu_{\min} = (m_B - m_A)c^2$. Therefore, from (23):

$$P_{\min} = \frac{1}{12\pi^2} \frac{((m_B - m_A)c^2)^4}{(\hbar c)^3} \quad (25)$$