1. Ideal gas of two-state atoms

Consider an ideal monoatomic gas made of N atoms each of which has only 2 internal states: a ground state and an excited state with energy gap equal to Δ. The gas is in a sealed container with no energy exchange with outside world. Initially, the gas is prepaired in such a way that all the atoms are in their ground state internally, but the gas is in thermal equilibrium with respect to kinetic motion of the atoms, characterized by temperature T_1. After some time, however, due to collisions, the internal degree of freedom of the atoms is also excited and thermalized.

a) Find the temperature of the gas, T_2, after the internal degree of freedom thermalizes. Assume $\Delta \ll kT_1$ and calculate the difference $T_2 - T_1$ up to order Δ. Does the temperature increase or decrease?

b) Find the change in the entropy of the gas, $S_2 - S_1$, after the complete thermalization. Assume $\Delta \ll kT_1$ and work up to order Δ. Does the entropy increase or decrease?

Hint: The entropy of the same gas *without* the internal degree of freedom is:

$$S_{\text{kin}} = \frac{3}{2} kN \ln T + (T \text{ independent terms}).$$

2. van der Waals equation of state

Consider van der Waals equation of state:

$$p = \frac{nkT}{1 - bn} - an^2;$$

where p is the pressure, n is the density and T is the temperature. a and b are positive coefficients.

a) There are 2 different types of isotherms $p(n)$, depending on the value of T. For some values of T the pressure is a monotonous function of the density, for other values it is not. Sketch these two types of the isotherms on the p vs n plot.

b) Now, on the same plot, sketch the curve where the derivative $(\partial p/\partial n)_T$ vanishes.

c) Shade the region of p and n where the system is thermodynamically unstable towards phase separation (not even metastable). Write the stability condition (inequality) you use.

d) Express the coordinates p_c, n_c and T_c of the critical point in terms of a and b.

3. Maxwell relations

Consider a rubber band of length L which is being stretched by external force f.

a) Write down the thermodynamic identity (1st law of thermodynamics) relating change in the internal energy dU to infinitesimal change in length dL, and to supplied heat TdS.

b) In one experiment the length of the band is fixed to $L = 1$m and the temperature of the band $T = 300$K is raised by a small amount $\Delta T = 3$K. This causes the force needed to maintain the length of the band to increase by the amount $\Delta f = 1.2$N. In another experiment, the band is stretched from L to $L + \Delta L$ at constant temperature T. As a result the band exchanges heat with the environment. What is the amount of this heat for $\Delta L = 2$cm? Is the heat released or absorbed by the band?

4. Cosmic microwave background

Cosmic microwave background (CMB, or relic) radiation is an isotropic radiation with a black body spectrum at temperature $T = 2.7$ K.

a) Find the density n of the CMB photons. How many relic photons are there on average inside a volume of space $V = 1\ \mathrm{cm}^3$?

b) Find the rate at which a ball of radius $R = 1$ cm is struck by relic photons.

You may find useful the following combination of constants:

$$\frac{k}{\hbar c} = 436.7\ \mathrm{K}^{-1}\mathrm{m}^{-1}$$

as well as this integral

$$\int_0^\infty dx \frac{x^2}{e^x - 1} = 2\zeta(3) = 2.404\ldots$$

5. Ultrarelativistic Fermi gas at $T = 0$

Matter inside a star can be compressed to such an extent that the Fermi energy of the electrons becomes much larger than their rest energy. Consider electron gas at $T = 0$ and given chemical potential μ, such that $\mu \gg m_e c^2$. In this regime Coulomb interaction is negligible.

a) What is the maximum momentum of an electron in such a gas?

b) What is the density of the electrons at this value of μ.

c) What is the total energy E of such a gas in a volume V containing N electrons. The answer should not contain μ.

d) What is the pressure P of the gas in terms of μ?

e) A nucleus of the substance, called A, can capture an electron and undergo a transformation:

$$\mathrm{A} + e^- \to \mathrm{B} + \nu \tag{1}$$

The mass of nucleus B is larger than the mass of A, therefore the reaction is energetically forbidden under normal conditions. However, at sufficiently high pressure $P > P_{\mathrm{min}}$ the reaction is allowed. Explain why and calculate P_{min}, given the masses m_A and m_B. Neglect the masses of electron and neutrino ($m_B - m_A \gg m_e$).