QUANTUM MECHANICS SOLUTIONS

(Preliminary Exam, Jan '05)

Q.1) a) Clearly, $\psi(x) = \begin{cases} Ax & \text{if } 0 \le x \le L/2 \\ A(L-x) & \text{if } L/2 \le x \le L \\ 0 & \text{elsewhere} \end{cases}$ where A is the normalization constant

$$\int_0^L |\psi(x)|^2 dx = 1 \Rightarrow |A|^2 \int_0^{L/2} x^2 dx = \frac{1}{2} \Rightarrow |A|^2 = \frac{12}{L^3} \Rightarrow A = \sqrt{\frac{12}{L^3}}\, e^{i\delta} \quad (\delta \text{ real})$$

take it $\delta = 0$

$$\Rightarrow \quad \psi(x) = \sqrt{\frac{12}{L^3}} \begin{cases} x & \text{if } 0 \le x \le L/2 \\ (L-x) & \text{if } L/2 \le x \le L \\ 0 & \text{elsewhere} \end{cases} \quad \text{is normalized}$$

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b) Particle in a box normalized $\hat{H}$-eigenstates are $\phi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ with eigenvalu[es] $E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$. $\{\phi_n(x)\}$ form a complete orthonormal basis $\Rightarrow$ If $\psi(x) = \sum_n c_n \phi_n(x)$

where $c_n = \int \psi(x)\, \phi_n^*(x)\, dx$, then $|c_n|^2$ is the probability of measuring E_n.

$$c_n = \frac{\sqrt{24}}{L^2} \left[\int_0^{L/2} x \sin\frac{n\pi x}{L} dx + \int_{L/2}^{L} (L-x) \sin\frac{n\pi x}{L} dx \right] = \frac{\sqrt{24}}{L^2} \left\{ \Big|_0^{L/2} \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L} - \left(\frac{L}{n\pi}\right) x \cos\frac{n\pi x}{L} \right.$$

$$\left. + L\left(\frac{-L}{n\pi}\right) \Big|_{L/2}^{L} \cos\frac{n\pi x}{L} - \Big|_{L/2}^{L} \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L} + \Big|_{L/2}^{L} \left(\frac{L}{n\pi}\right) x \cos\frac{n\pi x}{L} \right\} = \frac{\sqrt{24}}{L^2} \left\{ \frac{L^2}{n^2\pi^2} \sin\frac{n\pi}{2} \right.$$

$$- \cancel{\frac{L^2}{2\pi n} \cos\frac{n\pi}{2}} - \cancel{\frac{L^2}{n\pi} \cos n\pi} + \cancel{\frac{L^2}{n\pi} \cos\frac{n\pi}{2}} - \cancel{\frac{L^2}{n^2\pi^2} \underbrace{\sin n\pi}_{=0}} + \frac{L^2}{n^2\pi^2} \sin\frac{n\pi}{2} + \cancel{\frac{L^2}{n\pi} \cos \pi n} - \cancel{\frac{L^2}{2\pi n} \cos\frac{n\pi}{2}}$$

$$\Rightarrow C_n = \frac{\sqrt{24}}{L^2}\cdot\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} \mp\frac{4\sqrt{6}}{n^2\pi^2} & \text{if } n=\text{odd} \quad (+;\text{for } n=1,5,9,\ -\text{for } n=3,7,\ldots \\ 0 & \text{if } n=\text{even} \end{cases}$$

$\Rightarrow$ Probability of obtaining $E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2$ with n=odd is $\frac{96}{n^4\pi^4}$

with n=even is 0

In principle, all E_n's are possible (without regard to the form of $\psi(x)$), but since the given $\psi(x)$ is symmetric wrt $x=L/2$ (parity eigenstate), only symmetric $\phi_n(x)$ appear in its expansion. Symmetric ϕ_n's are given with n=odd

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(c) $$\langle E\rangle = \sum_n E_n P_n = \sum_{k=0}^{\infty}\frac{\hbar^2\pi^2(2k+1)^2}{2mL^2}\cdot\frac{96}{(2k+1)^4\pi^4} = \frac{48\hbar^2}{mL^2\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \underline{\underline{\frac{6\hbar^2}{mL^2}}}$$

or

$$\langle E\rangle = \int_0^L \psi^*(x)\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right]\psi(x) = \frac{-\hbar^2}{2m}(-2A)\psi\left(x=\frac{L}{2}\right) = \frac{\hbar^2}{2m}2\cdot\sqrt{\frac{12}{L^3}}\sqrt{\frac{12}{L^3}}\cdot\frac{L}{2} = \underline{\underline{\frac{6\hbar^2}{mL^2}}} \checkmark$$

δ-function picks ψ at $x=\frac{L}{2}$

(d) The probability of measuring a momentum value in the Δp infinitesimal vicinity of $p=$ is given by $|a(p=0)|^2\Delta p$, where $a(p)$ is the momentum wavefunction

$$a(p) = \frac{1}{\sqrt{2\pi\hbar}}\int_0^L e^{-i\frac{px}{\hbar}}\psi(x)\,dx \Rightarrow a(0) = \frac{A}{\sqrt{2\pi\hbar}}\left[\int_0^{L/2}x\,dx + \int_{L/2}^{L}(L-x)\,dx\right] = \frac{A}{\sqrt{2\pi\hbar}}\left[\frac{L^2}{8}+\frac{L^2}{8}\right]$$

$$\Rightarrow |a(0)|^2\Delta p = \frac{A^2}{2\pi\hbar}\,\frac{L^4}{16}\cdot\frac{\hbar}{100L} = \frac{12}{L^3}\,\frac{1}{2\pi\hbar}\,\frac{\hbar L^4}{1600L} = \frac{3}{800\pi} \Rightarrow \frac{3}{800\pi}\times 10{,}000 \simeq \underline{\underline{12}}$$

Approx. # of measurement p is measured to be between 0 &

Q.2) In the $|1\rangle, |2\rangle$ basis, H is represented by the matrix $H = \begin{pmatrix} \cdots & \cdots \\ -3\hbar\omega & 2\hbar\omega \end{pmatrix}$

$|\alpha\rangle$ & $|\beta\rangle$ are eigenvectors of $\hat{H}$ with E_1, E_2 eigenvalues $(E_1 < E_2)$

Find them $\Rightarrow$ $\det \begin{pmatrix} 10\hbar\omega - \lambda & -3\hbar\omega \\ -3\hbar\omega & 2\hbar\omega - \lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 12\hbar\omega\lambda + 11\hbar^2\omega^2 = 0 \Rightarrow \lambda_1 = E_1 = \hbar\omega$

$\lambda_2 = E_2 = 11\hbar\omega$

$$\hat{H}|\alpha\rangle = \hbar\omega|\alpha\rangle \Rightarrow |\alpha\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{10}}|1\rangle + \frac{3}{\sqrt{10}}|2\rangle$$

$$|\beta\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \frac{3}{\sqrt{10}}|1\rangle - \frac{1}{\sqrt{10}}|2\rangle$$

In the $|\alpha\rangle, |\beta\rangle$ basis, $\hat{A}$ is represented by the matrix $A = \begin{pmatrix} 0 & -2ia_0 \\ 2ia_0 & -3a_0 \end{pmatrix}$

The eigenvalues of $\hat{A}$ are $\det \begin{vmatrix} -\lambda & -2ia_0 \\ 2ia_0 & -\lambda - 3a_0 \end{vmatrix} = 0 \Rightarrow \lambda_1 = a_0 \rightarrow \frac{2}{\sqrt{5}}|\alpha\rangle + \frac{i}{\sqrt{5}}|\beta\rangle$

$\underset{\uparrow\ \text{Eigenvalues}}{\lambda_2 = -4a_0} \hookrightarrow \underbrace{\frac{1}{\sqrt{5}}|\alpha\rangle - \frac{2i}{\sqrt{5}}|\beta\rangle}_{\text{Eigenvectors expre in } |\alpha\rangle, |\beta\rangle \text{ basis}}$

PART I:

a) $\hat{A}$-measurement yielding the largest possible value (must have found a_0, since $a_0 > 0$) collapses $|\psi(0)\rangle$ to the eigenstate of $\hat{A}$ with eigenvalue a_0 (Reduction & measurement postulates)

$$\Rightarrow |\psi(0)\rangle = \frac{2}{\sqrt{5}}|\alpha\rangle + \frac{i}{\sqrt{5}}|\beta\rangle$$

Since $|\psi(0)\rangle$ has already been expressed in terms of $\hat{H}$-eigenstates, it is trivia to write its time evolution,

$$|\psi(t)\rangle = \frac{2}{\sqrt{5}}|\alpha\rangle e^{-i\omega t} + \frac{i}{\sqrt{5}}|\beta\rangle e^{-i11\omega t}$$

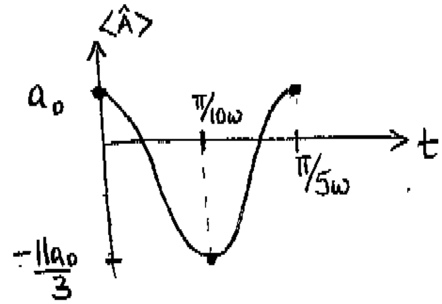
$\mathbb{P}$ of measuring a_0 again means calculating $|\langle\psi(0)|\psi(t)\rangle|$

$$\mathbb{P}(t)=\left|\underbrace{\begin{pmatrix}\frac{2}{\sqrt5} & \frac{-i}{\sqrt5}\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt5}e^{-i\omega t}\\ \frac{i}{\sqrt5}e^{-i11\omega t}\end{pmatrix}}_{\text{all expressed wrt } |\alpha\rangle,|\beta\rangle \text{ basis}}\right|^2=\left|\frac{4}{5}e^{-i\omega t}+\frac{1}{5}e^{-i11\omega t}\right|^2=\underline{\underline{\frac{17}{25}+\frac{8}{25}\cos 10\omega t}}$$

(b)

$$\langle\hat{A}\rangle(t)=\underbrace{\begin{pmatrix}\frac{2}{\sqrt5}e^{i\omega t} & \frac{-i}{\sqrt5}e^{+i11\omega t}\end{pmatrix}}_{\langle\psi(t)|}\underbrace{a_0\begin{pmatrix}0 & -2i\\ 2i & -3\end{pmatrix}}_{\hat{A}}\underbrace{\begin{pmatrix}\frac{2}{\sqrt5}e^{-i\omega t}\\ \frac{i}{\sqrt5}e^{-i11\omega t}\end{pmatrix}}_{|\psi(t)\rangle}=\frac{a_0}{5}\begin{pmatrix}2e^{i\omega t} & -ie^{i11\omega t}\end{pmatrix}\begin{pmatrix}2e^{-i11\omega t}\\ 4ie^{-i\omega t}-3ie^{-i11\omega t}\end{pmatrix}$$

$$=\frac{a_0}{5}\left(4e^{-i10\omega t}+4e^{i10\omega t}-3\right)=\frac{(8\cos 10\omega t-3)a_0}{5}$$



PART II: Let the probability of obtaining a_0 be $|c_1|^2 \Rightarrow$ probability of obtaining $-4a_0$ will be $(1-|c_1|^2) \Rightarrow$

$$\langle A\rangle=|c_1|^2a_0+(1-|c_1|^2)-4a_0=-\frac{a_0}{4}\Rightarrow|c_1|^2=\frac{3}{4}$$

$\Rightarrow |\psi\rangle=\frac{\sqrt3}{2}|\gamma\rangle+\frac{e^{i\delta}}{2}|\delta\rangle$ where δ is an arbitrary phase factor and $|\gamma\rangle$ and $|\delta\rangle$ are eigenvectors of $\hat{A}$ with eigenvalues a_0 & $-4a_0$, respectively.

From diagonalization of $\hat{A}$ before, we have $|\gamma\rangle=\frac{2}{\sqrt5}|\alpha\rangle+\frac{i}{\sqrt5}|\beta\rangle$

and $|\delta\rangle=\frac{1}{\sqrt5}|\alpha\rangle-\frac{2i}{\sqrt5}|\beta\rangle$

$$\Rightarrow|\psi\rangle=\frac{\sqrt3}{2}\left(\frac{2}{\sqrt5}|\alpha\rangle+\frac{i}{\sqrt5}|\beta\rangle\right)+\frac{e^{i\delta}}{2}\left(\frac{1}{\sqrt5}|\alpha\rangle-\frac{2i}{\sqrt5}|\beta\rangle\right)=\underbrace{\left(\sqrt{\frac35}+\frac{e^{i\delta}}{2\sqrt5}\right)}_{C}|\alpha\rangle+\underbrace{\left(\frac{i\sqrt3}{2\sqrt5}-\frac{ie^{i\delta}}{\sqrt5}\right)}_{D}|\beta\rangle$$

Q.3) Normalize $\psi_g(r)$ first

$$A^2\int|\psi_g(\vec{r})|^2 d^2\vec{r}=1 \Rightarrow A^2\cdot 2\pi\int_0^\infty r e^{-2\alpha r}dr=1 \Rightarrow 2\pi A^2\frac{1}{(2\alpha)^2}=1 \Rightarrow \underline{\underline{A=\alpha\sqrt{\frac{2}{\pi}}}}$$

$$H=-\frac{\hbar^2}{2\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\right)-\frac{e^2}{4\pi\epsilon_0 r}$$

$$\langle KE\rangle=\frac{-\hbar^2A^2}{2\mu}\cdot\underbrace{2\pi}_{\text{angular integration}}\left[\int_0^\infty e^{-\alpha r}(\alpha^2)e^{-\alpha r}r\,dr+\int_0^\infty e^{-\alpha r}(-\alpha)e^{-\alpha r}dr\right]$$

$$=-\frac{\hbar^2A^2\pi}{\mu}\left[\alpha^2\cdot\frac{1}{(2\alpha)^2}-\alpha\frac{1}{2\alpha}\right]=\frac{+\hbar^2A^2\pi}{4\mu}=\underline{\underline{\frac{\hbar^2\alpha^2}{2\mu}}}$$

$$\langle V\rangle=\frac{-e^2 2\pi}{4\pi\epsilon_0}A^2\int_0^\infty e^{-2\alpha r}dr=\frac{-e^2}{8\pi\epsilon_0\alpha}2\pi\frac{2\alpha^2}{\pi}=\underline{\underline{-\frac{e^2\alpha}{2\pi\epsilon_0}}}$$

$$\langle H\rangle=\frac{\hbar^2\alpha^2}{2\mu}-\frac{e^2\alpha}{2\pi\epsilon_0}\qquad\frac{d\langle H\rangle}{d\alpha}=0\Rightarrow\frac{\hbar^2\alpha}{\mu}-\frac{e^2}{2\pi\epsilon_0}\Rightarrow\alpha_{min}=\frac{\mu e^2}{2\pi\hbar^2\epsilon_0}$$

$$\Rightarrow E_{min}=\frac{\hbar^2}{2\mu}\frac{\mu^2e^4}{4\pi^2\hbar^4\epsilon_0^2}-\frac{e^2}{2\pi\epsilon_0}\frac{\mu e^2}{2\pi\hbar^2\epsilon_0}=-\frac{\mu e^4}{8\pi^2\hbar^2\epsilon_0^2}=\underline{\underline{-54.4\,eV}}$$

4 times larger than the energy of the 3D H-atom.

3b) Since $\psi_{1,0}(r,\phi)$ & $\psi_{2,0}(r,\phi)$ have the same angular part (no ϕ-dependence), their orthogonality to each other is established through the orthogonality of $R_{1,0}(r)$ & $R_{3,0}(r)$ (hence the proposed form)

$$\langle\psi_{1,0}|\psi_{2,0}\rangle=0 \Rightarrow \int_0^\infty dr\, e^{-\alpha r}(1-\beta r)e^{-\gamma r}\, r = \int_0^\infty dr\, r\, e^{-(\alpha+\gamma)r} - \beta\int_0^\infty dr\, r^2 e^{-(\alpha+\beta)r}=0 \Rightarrow$$

$$\frac{1}{(\alpha+\gamma)^2}-\frac{2\beta}{(\alpha+\gamma)^3}=0 \Rightarrow 1-\frac{2\beta}{\alpha+\gamma}=0 \Rightarrow \underline{\underline{\beta=\frac{\alpha+\gamma}{2}}}$$

3c) $H_1=-eE_0x=-eE_0 r\cos\phi=\frac{-eE_0 r}{2}\left(e^{i\phi}+e^{-i\phi}\right)$

Degenerate perturbation $\Rightarrow$ need to calculate $W_{mm'}=\langle\psi_{2,m}|H_1|\psi_{2,m'}\rangle$. Since $H_1\propto\left(e^{i\phi}+e^{-i\phi}\right)$ only $m=m'\mp1$ matrix elements are non-vanishing.

$$W \rightarrow \begin{array}{c|ccc} m \downarrow,\ m' \rightarrow & -1 & 0 & 1 \\ \hline -1 & 0 & B & 0 \\ 0 & B^* & 0 & B \\ 1 & 0 & B^* & 0 \end{array}$$

where $B=-\frac{eE_0}{2}\int_0^\infty r^2 R_{20}(r)R_{21}(r)\,dr\,\frac{1}{2\pi}\underbrace{\int_0^{2\pi}\left(\cancel{e^{i2\phi}}+1\right)d\phi}_{2\pi}$

$$=\frac{-eE_0 I}{2} \quad \Rightarrow B=B^*$$

Diagonalize W

$$\begin{vmatrix} -\lambda & B & 0 \\ B & -\lambda & B \\ 0 & B & -\lambda \end{vmatrix}=0 \Rightarrow -\lambda\left(\lambda^2-B^2\right)-B\left(-\lambda B\right)=0 \Rightarrow \lambda\left(\lambda^2-2B^2\right)=0$$

$$\Rightarrow \lambda_1=0 \ \& \ \lambda_{2,3}=\mp\sqrt{2}\,B$$

The shifts are $0, \mp\frac{eE_0 I}{\sqrt{2}}$

Q.4) a) $Y_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi} \Rightarrow Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{-i\phi}$, $x = r\sin\theta\cdot\left(\frac{e^{i\phi}+e^{-i\phi}}{2}\right)$

$$\Rightarrow x = \sqrt{\frac{2\pi}{3}}\left(Y_{1,-1} - Y_{1,1}\right) r \quad . \quad \text{Also} \quad Y_{00} = \frac{1}{\sqrt{4\pi}} \Rightarrow 1 = \sqrt{4\pi}\, Y_{0,0}$$

$$\psi(\vec{r}) = C \underbrace{\left[\sqrt{4\pi}\, Y_{00} + \sqrt{\frac{2\pi}{3}}\left(Y_{1,-1} - Y_{1,1}\right)\right]}_{g(\theta,\phi)} f(r)$$

Since $f(r)$ is normalized over r, for $\psi(\vec{r})$ to be normalized $g(\theta,\phi) = \sum_{\ell,m} c_{\ell m} Y_{\ell m}(\theta,\phi)$ should be normalized. Since $\{Y_{\ell m}\}$ complete & orthonormal $\Rightarrow$

$$\sum_{\ell m} |c_{\ell m}|^2 = 1 \Rightarrow 4\pi |c|^2 + 2 \otimes \frac{2\pi}{3}|c|^2 = 1 \Rightarrow \underline{\underline{C = \sqrt{\frac{3}{16\pi}}\, e^{i\delta}}}$$

(b) $$g(\theta,\phi) = \sqrt{\frac{3}{4}}\, Y_{00} + \frac{1}{2\sqrt{2}}\, Y_{1,-1} - \frac{1}{2\sqrt{2}}\, Y_{1,1}$$

$$\langle \vec{L}^2 \rangle = \sum_{\ell m} \ell(\ell+1)\hbar^2 |c_{\ell m}|^2 = 0(0+1)\hbar^2 \cdot \frac{3}{4} + 1(1+1)\hbar^2 \frac{1}{4} = \underline{\underline{\frac{\hbar^2}{2}}}$$

(c) $\phi(\vec{r}) = D r^2 \sin\theta \cos\phi\, e^{-\alpha r} \Rightarrow R(r) = r^2 e^{-\alpha r}$ (radial part)

and $\phi(\vec{r})$ is an $\vec{L}^2$ eigenstate with $\ell = 1$

From the wording of the problem $\phi(\vec{r})$ is also a Hamiltonian eigenstate $\Rightarrow R(r)$ satisfies radial S.E. with $\ell = 1$. Need $\frac{dR}{dr}$ & $\frac{d^2R}{dr^2} \Rightarrow$ • $\frac{dR}{dr} = -\alpha r^2 e^{-\alpha r} + 2r e^{-\alpha r}$

• $\frac{d^2R}{dr^2} = \alpha^2 r^2 e^{-\alpha r} - 2\alpha r e^{-\alpha r} + 2e^{-\alpha r} - 2\alpha r e^{-\alpha r}$

$$-\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}-\frac{l(l+1)}{r^2}R\right)+V(r)R=ER \quad \text{, plug in} \Rightarrow$$

$$-\frac{\hbar^2}{2m}\left(\alpha^2r^2-4\alpha r\cancel{+2}-2\alpha r+4\cancel{-2}\right)e^{-\alpha r}+V(r)r^2e^{-\alpha r}=Er^2e^{-\alpha r}$$

Divide by $r^2e^{-\alpha r} \Rightarrow -\frac{\hbar^2}{2m}\left[\frac{4}{r^2}-\frac{6\alpha}{r}+\alpha^2\right]=E-V(r)$. In the limit $r\to\infty$,

this becomes $\underline{\underline{E=-\frac{\hbar^2\alpha^2}{2m}}}$ $\Rightarrow$ $\underline{\underline{V(r)=\frac{\hbar^2}{2m}\left(\frac{4}{r^2}-\frac{6\alpha}{r}\right)}}$

(d) $m_1=1$ & $m_2=-\frac{1}{2}$ can come from $\left(m_1+m_2=M=\frac{1}{2}\right)$ $|L=\frac{3}{2},M=\frac{1}{2}\rangle$ and $|L=\frac{1}{2},M=\frac{1}{2}\rangle$ coupled states. From the given CG-coefficients, only 2 are relevant $\Rightarrow$

$$|\tfrac{3}{2},\tfrac{1}{2}\rangle=\sqrt{\tfrac{2}{3}}\,|0\rangle_1|\tfrac{1}{2}\rangle_2+\sqrt{\tfrac{1}{3}}\,|1\rangle_1|-\tfrac{1}{2}\rangle_2$$

$$\underbrace{|\tfrac{1}{2},\tfrac{1}{2}\rangle}_{|L,M\rangle}=-\sqrt{\tfrac{1}{3}}\,|0\rangle_1|\tfrac{1}{2}\rangle_2+\sqrt{\tfrac{2}{3}}\,|1\rangle_1|-\tfrac{1}{2}\rangle_2$$

$|m_1\rangle_1|m_2\rangle_2$

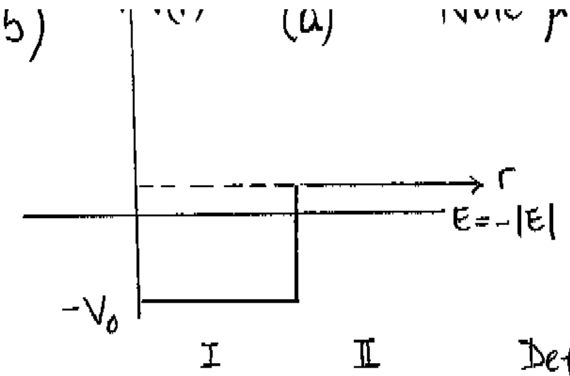
We need the inverse transformation, but we know $O^{-1}=O^T$

$$\Rightarrow |0\rangle_1|\tfrac{1}{2}\rangle_2=\sqrt{\tfrac{2}{3}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle-\frac{1}{\sqrt{3}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

$$\longrightarrow \underbrace{|1\rangle_1|-\tfrac{1}{2}\rangle_2}_{\text{given state}}=\frac{1}{\sqrt{3}}\,|\tfrac{3}{2},\tfrac{1}{2}\rangle+\sqrt{\tfrac{2}{3}}\,\underbrace{|\tfrac{1}{2},\tfrac{1}{2}\rangle}_{\to\ \text{state with } j=1/2\ \left(|\vec{J}|=\frac{\hbar\sqrt{3}}{4}\right)}$$

$$\Rightarrow P \text{ of } j=\tfrac{1}{2}=\left|\sqrt{\tfrac{2}{3}}\right|^2=\underline{\underline{\tfrac{2}{3}}}$$

Q.5) (a) Note $\mu = \frac{\ldots}{2}$



Region I $\Rightarrow -\frac{\hbar^2}{2\mu}\frac{d^2u_1}{dr^2} - V_0 u_1(r) = -|E|u_1(r)$

Region II $\Rightarrow -\frac{\hbar^2}{2\mu}\frac{d^2u_2}{dr^2} = -|E|u_1(r)$

Define $k^2 = \frac{2\mu(V_0 - |E|)}{\hbar^2}$ and $q^2 = \frac{2\mu|E|}{\hbar^2}$

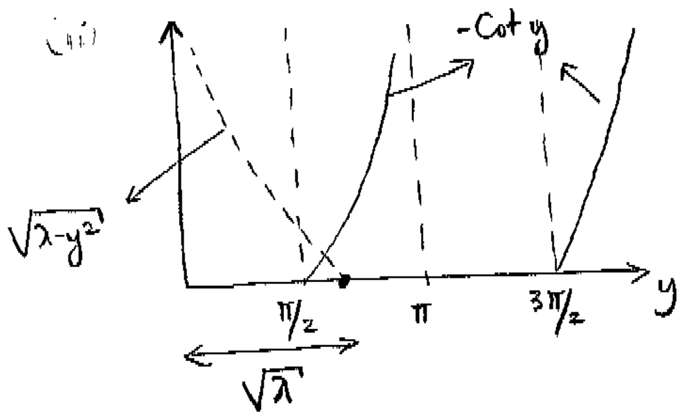
$u_1(r) = A\sin kr$ (regular at $r=0$)

$u_2(r) = Be^{-qr}$ (regular at $r=\infty$)

$u_1(R) = u_2(R)$, $u_1'(R) = u_2'(R)$ $\Rightarrow$ $A\sin kR = Be^{-qR}$, $kA\cos kR = -Bqe^{-qR}$

$\Rightarrow -\cot kR = \frac{q}{k} = \frac{\sqrt{|E|}}{\sqrt{V_0 - |E|}}$ or $y = kR$ & $\lambda = \frac{2\mu V_0 R^2}{\hbar^2}$ $\Rightarrow -\cot y = \frac{\sqrt{\lambda - y^2}}{y}$

(b) (i) Plot $-\cot y$ & $\frac{\sqrt{\lambda - y^2}}{y}$ on the same graph. Intersection(s) $\rightarrow$ bound state



For there to be a bound state $\sqrt{\lambda} \geq \pi/2$

$\Rightarrow \frac{2\mu V_{0,min} R^2}{\hbar^2} = \frac{\pi^2}{4} \Rightarrow \underline{\underline{V_{0,min} = 24.4\ \text{MeV}}}$

(ii.) Probably, the hint means $-\cot y = \frac{\sqrt{\lambda - y^2}}{y}$ is satisfied with the given parameters for $y = 1.824$. Let's check: $+0.259 = \frac{\sqrt{\lambda - (1.824)^2}}{1.824} \Rightarrow \lambda = 3.55 \Rightarrow \underline{\underline{V_0 = 35.2\ \text{MeV}}}$

Is $\frac{\sqrt{2.2}}{\sqrt{35.2 - 2.2}} = 0.259$ ✓ yes

(c) Since the proton & neutron are both $s=1/2$ particles, the total spin of deuteron ranges in integer steps from $|s_p - s_n|$ to $s_p + s_n$, i.e. 0 to 1 $\Rightarrow$ $\underline{\underline{s_{deut} = 0 \text{ or } 1}}$

(d) To end up with $j_D = 1$, [since $\vec{J} = \vec{L} + \vec{S}$, j_D ranges from $|l-s|$ to $l+s$]

If $s_d = 0 \Rightarrow l = 1$

if $s_d = 1 \Rightarrow l$ can be 0 or 1 or 2

i.e. $(l=0, s=1)$ can give $j = \underline{\underline{1}}$ ✓

$(l=1, s=1)$ can give $j = 0, \underline{\underline{1}}, 2$ ✓

$(l=2, s=1)$ can give $j = \underline{\underline{1}}, 2, 3$ ✓

The possible (l,s) combinations are

$(1,0)$ $\quad {}^0P_1$

$(0,1)$ $\quad$ or $\quad {}^3S_1$

$(1,1)$ $\quad {}^3P_1$

$(2,1)$ $\quad {}^3D_1$

(e) $|\phi_1\rangle \rightarrow (0,1)$ [i.e. ${}^3S_1$]

Parity of (l,s) is $(-1)^l$. Parity of $(0,1)$ is even, only the parity of $(2,1)$ [${}^3D_1$] of the remaining ones is even $\Rightarrow$ $\underline{\underline{|\phi_2\rangle = (l=2, s=1) \text{ i.e } {}^3D_1}}$

(f) $\langle \phi_1 | \mu | \phi_1 \rangle = \langle \mu \rangle_{{}^3S_1} = \left[0.5 + 0.095 \times (2 + 1.2 - 0)\right] \mu_N = 0.88 \mu_N$

$\langle \phi_2 | \mu | \phi_2 \rangle = \langle \mu \rangle_{{}^3D_1} = \left[0.5 + 0.095 \times (2 + 2 - 2\times 3)\right] \mu_N = 0.31 \mu_N$

If $|\psi_g\rangle = A|\phi_1\rangle + B|\phi_2\rangle \Rightarrow A^2 + B^2 = 1$ (Normalization)

$0.88\mu_N A^2 + 0.31\mu_N B^2 = 0.86\mu_N \quad (\langle \mu \rangle_g = 0.86 \mu_N)$

Solve for A & B $\Rightarrow$ $A^2 \simeq 0.96$, $A \simeq 0.98$, $B \simeq 0.19$ $\Rightarrow |\psi_g\rangle \simeq 0.98 |{}^3S_1\rangle + 0.19 |{}^3D_1\rangle$