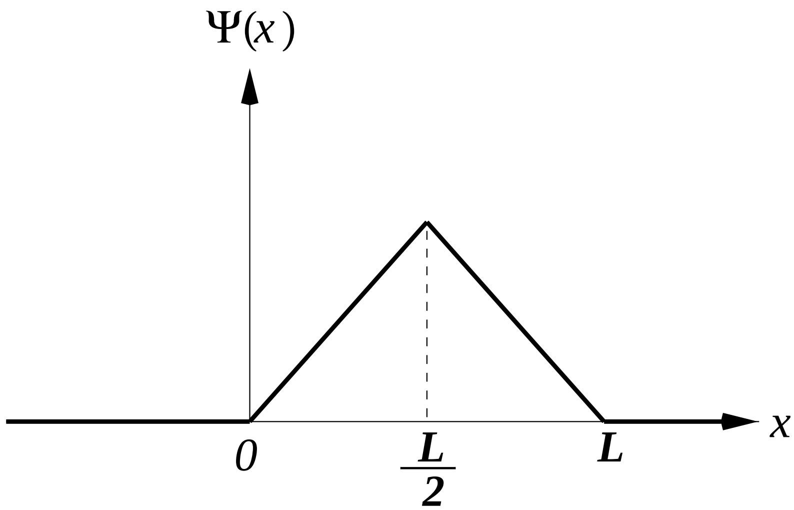
1. A particle of mass m is in a one-dimensional infinite potential well of width L. That is, the potential is given by

$$V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}$$

The particle has been prepared to have the wavefunction $\psi(x)$ as shown in the following graph (All parts of the plot are straight lines).



(a) Write down the normalized wavefunction $\psi(x)$ as a piecewise function of x.

(b) If an energy measurement is made on this particle, what values can be obtained from the measurement, and what are the corresponding probabilities? Explain why certain energy values, which are in general possible in an energy measurement on a particle in an infinite potential well, are not allowed for this wavefunction.

(c) Calculate the expectation value of the energy of this particle. One possible way of solving this part requires the following expression:

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

(d) The spread in the momentum of this particle will be of the order of $\sim \hbar/L$. In that sense, a momentum interval of $\Delta p = \frac{\hbar}{100L}$ can be considered to be infinitesimal.

10,000 identical particles, all of which have the same wavefunction given above are in separate ∞−potential boxes of width L. Momentum measurements are made on all 10,000 particles. Approximately, in how many experiments will momentum values in the $\Delta p = \hbar/100L$ infinitesimal vicinity of $p = 0$ be measured?

2. Consider a two-state quantum system. In the orthonormal and complete set of basis vectors $|1\rangle$ and $|2\rangle$, the Hamiltonian operator for the system is represented by ($\omega > 0$)

$$\hat{H} = 10\hbar\omega|1\rangle\langle 1| - 3\hbar\omega|1\rangle\langle 2| - 3\hbar\omega|2\rangle\langle 1| + 2\hbar\omega|2\rangle\langle 2|$$

Let us consider another complete and orthonormal basis $|\alpha\rangle$, $|\beta\rangle$, such that $\hat{H}|\alpha\rangle = E_1|\alpha\rangle$, and $\hat{H}|\beta\rangle = E_2|\beta\rangle$ (with $E_1 < E_2$). Let the action of operator $\hat{A}$ on the $|\alpha\rangle$, $|\beta\rangle$ basis vectors be given as

$$\begin{aligned} \hat{A}|\alpha\rangle &= 2ia_0|\beta\rangle \\ \hat{A}|\beta\rangle &= -2ia_0|\alpha\rangle - 3a_0|\beta\rangle \end{aligned}$$

where $a_0 > 0$ is real. Answer the next two *independent* parts based on the information given above:

PART I:

Suppose an $\hat{A}-$measurement is carried out at $t = 0$ on an arbitrary state and the largest possible value is obtained.

(a) Calculate the probability $P(t)$ that another measurement made at time t will yield the same value as the one measured at $t = 0$.

(b) Calculate the time dependence of the expectation value $\langle\hat{A}\rangle$. Plot $\langle\hat{A}\rangle(t)$ as a function of time. What is the minimum value of $\langle\hat{A}\rangle$? At what time is it first achieved?

PART II

Suppose that the average value obtained from a large number of $\hat{A}-$measurements on identical quantum states at a given time is $-a_0/4$.

(c) Construct the most general normalized state vector (just before the $\hat{A}-$measurement) for your system consistent with this information in Dirac notation using the $|\alpha\rangle$, $|\beta\rangle$ basis. Express your answer as

$$|\psi\rangle = C|\alpha\rangle + D|\beta\rangle$$

3. If the motion of an electron (of charge $-e$) around a nucleus (of charge $+e$ and mass $m \gg m_e$) is constrained in a plane by certain boundary conditions, we can call such a system a "two-dimensional (2D) hydrogen atom". In this problem, you will examine some of the properties of the 2D hydrogen atom bound states using polar coordinates (r, ϕ). The potential for the system is

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

(a) In analogy with the three dimensional hydrogen atom, let us propose the trial wavefunction

$$\psi_g(r, \phi) = Ae^{-\alpha r}$$

for the ground state of the 2D hydrogen atom. Using the variational principle with this wavefunction, calculate the ground state energy of the 2D hydrogen atom in eV, and compare your answer to the ground state energy of the 3D hydrogen atom.

It turns out that the above trial wavefunction is indeed the correct ground state wavefunction of the 2D hydrogen atom. When the 2D hydrogen atom is solved analytically, the ground state corresponds to quantum numbers $n = 1$, $l = 0$, and is non-degenerate. The first excited state, on the other hand, is three-fold degenerate with wavefunctions given by $n = 2$, $l = 0, \pm 1$

$$\begin{aligned}
\psi_{2,0}(r, \phi) &= \frac{1}{\sqrt{2\pi}} R_{2,0}(r) \\
\psi_{2,\pm 1}(r, \phi) &= \frac{1}{\sqrt{2\pi}} R_{2,1}(r) e^{\pm i\phi}
\end{aligned}$$

where $R_{2,0}(r)$ and $R_{2,1}(r)$ are real radial functions.

(b) In analogy with the 3D hydrogen atom, the radial part of the energy eigenfunction for the $n = 2$, $l = 0$ state can be proposed to be of the form

$$R_{2,0}(r, \phi) = B(1 - \beta r)e^{-\gamma r}$$

Express β in terms of α and γ (You do not need to evaluate B or γ, or use the calculated value of α. All you are asked is a simple relationship between α, β, γ).

(c) A small electric field of magnitude E_0 is applied to the 2D hydrogen atom in the $x-$direction (in the plane). Calculate the shifts in the first excited state energy to first order in E_0. Your answers should be in terms of e, E_0, and I, where I is defined as the following integral:

$$I = \int_0^\infty dr\ r^2 R_{2,0}(r) R_{2,1}(r)$$

4. An electron in a central (spherically symmetric) potential $V(r)$ is described by the normalized wavefunction

$$\psi(\mathbf{r}) = C\left(1 + \frac{x}{r}\right) f(r)$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and the function $f(r)$ is such that

$$\int_0^\infty r^2 |f(r)|^2 dr = 1$$

(a) Calculate the constant C.

(b) Calculate the expectation value of the electron's angular momentum square $\langle \hat{\mathbf{L}}^2 \rangle$.

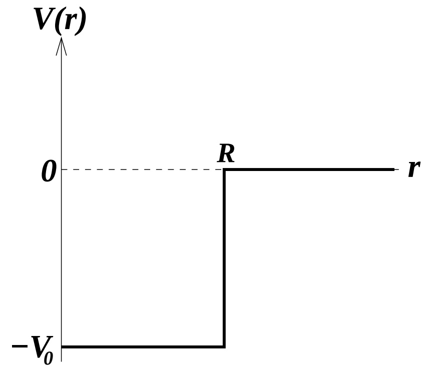
(c) (This part is independent of the first two parts) Consider a slightly different wavefunction $\phi(\mathbf{r})$ with a particular choice of $f(r)$ as follows (D is just a normalization constant, which you are not required to calculate):

$$\phi(\mathbf{r}) = Dxre^{-\alpha r}$$

If you are told that an energy measurement on the electron described by $\phi(\mathbf{r})$ is sure to yield (with 100 % certainty) a particular energy value E, calculate this E and the central potential $V(r)$ in terms of $\hbar$, α, and m (mass of the electron). Assume $V(r \to \infty) = 0$.

(d) (This part is independent of the rest) Simultaneous measurements of the $z-$components of the electron's orbital and spin angular momenta have yielded values of $+\hbar$ and $-\hbar/2$. If a measurement of the electron's total angular momentum ($\mathbf{J} = \mathbf{L} + \mathbf{S}$) is made immediately following these measurements, what is the probability of obtaining a value of $\frac{\hbar\sqrt{3}}{2}$ for the magnitude of $\mathbf{J}$?

5. Most properties of the deuteron (a bound state of a proton and a neutron) can be well described by considering a spherically symmetric square well potential of depth V_0 for the relative coordinate $r = |\mathbf{r}_p - \mathbf{r}_n|$. The potential $V(r)$ is shown below. Take $R = 2.05 \times 10^{-15}$ m. For convenience, you can assume the masses of the proton and the neutron to be same $m = 1.67 \times 10^{-27}$ kg.



(a) Write down the solutions of the radial Schrödinger equation for $u(r) = rR(r)$ assuming the existence of a bound state of energy $E = -|E|$ in the angular momentum state $l = 0$. Match the boundary conditions, and obtain a transcendental equation involving a trigonometric function.

(b) **(i)** What is the minimum depth $V_{0,\text{min}}$ of the potential for a bound state to exist? Express your answer in MeV. **(ii)** Calculate an accurate value (in MeV) for the actual depth V_0 using the information that the experimentally measured binding energy of the deuteron is $|E| = 2.2$ MeV. (**Hint:** $\cot 1.824 \approx -0.259$)

The rest of this problem is independent of the first two parts.

(c) Write down the possible total spin quantum numbers s of the deuteron ($\mathbf{S} = \mathbf{S}_p + \mathbf{S}_n$).

(d) It is known experimentally that the deuteron is in a total angular momentum ($\mathbf{J}$=$\mathbf{L}$+$\mathbf{S}$) state with $j = 1$. Here $\mathbf{L}$ and $\mathbf{S}$ denote the total orbital and spin angular momentum vectors, respectively, of the deuteron. Based on this information and your result in part **(c)**, find the possible orbital quantum numbers l for the deuteron with their corresponding s values. Write down all the possible combinations as (l, s).

(e) In one of the possible combinations above you should have $l = 0$. Let us call this state $|\phi_1\rangle$. If we were to write the ground state deuteron wavefunction $|\psi_g\rangle$ as a superposition of $|\phi_1\rangle$ and other (l, s) states, *only one* of the other (l, s) states could appear in the superposition. Explain why. (**Hint:** Strong force conserves parity).

(f) Let us call this other state $|\phi_2\rangle$ so that the normalized $|\psi_g\rangle = A|\phi_1\rangle + B|\phi_2\rangle$. The expectation value of the deuteron magnetic moment in the state labeled by (l, s) can be shown to be (with $j = 1$)

$$\langle \mu \rangle_{l,s,j=1} = [0.5 + 0.095 \times (2 + s(s+1) - l(l+1))]\, \mu_N$$

where μ_N is the nuclear magneton. If the measured magnetic moment of the deuteron is $0.86\mu_N$, calculate the coefficients A and B, and show that taking $|\psi_g\rangle$ to be entirely an $l = 0$ state (i.e. $|\phi_1\rangle$) is a rather good approximation.

Some Possibly Useful Information

- For a particle in a one dimensional $\infty-$potential box of width L, the normalized eigenstates of the Hamiltonian are $\phi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$.

- $Y_{0,0} = \dfrac{1}{\sqrt{4\pi}} \quad Y_{1,1} = -\sqrt{\dfrac{3}{8\pi}}\sin\theta e^{i\phi} \quad Y_{1,0} = \sqrt{\dfrac{3}{4\pi}}\cos\theta$

$$Y_{2,2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi} \quad Y_{2,1} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \quad Y_{2,0} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

- $Y_{l,-m} = (-1)^m Y^*_{lm}$

- $$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}$$

- $\displaystyle\int_0^\infty dx\ x^n\ e^{-ax} = \frac{n!}{a^{n+1}} \quad n \geq 0,\quad \text{integer}$

- $\displaystyle\int dx\ x\sin(Ax) = \frac{\sin(Ax)}{A^2} - \frac{x\cos(Ax)}{A}$

- Laplacian in 2D polar coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}$$

- 3D radial Schrödinger equations for the radial wavefunction $R(r)$ and $u(r) = rR(r)$:

$$\begin{aligned} -\frac{\hbar^2}{2m}\left(\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} - \frac{l(l+1)}{r^2}R(r)\right) + V(r)R(r) &= ER(r) \\ -\frac{\hbar^2}{2m}\frac{d^2u(r)}{dr^2} + \left(V(r) + \frac{l(l+1)\hbar^2}{2mr^2}\right)u(r) &= Eu(r) \end{aligned}$$

- For $l_1 = 1$, $l_2 = 1/2$, the $|L, M\rangle$ in the coupled (total angular momentum) representation are given in terms of the uncoupled basis vectors $|m_1\rangle_1|m_2\rangle_2$ as:

$$\begin{aligned} |3/2, 3/2\rangle &= |1\rangle_1|1/2\rangle_2 \\ |3/2, 1/2\rangle &= \sqrt{2/3}|0\rangle_1|1/2\rangle_2 + \sqrt{1/3}|1\rangle_1|-1/2\rangle_2 \\ |3/2, -1/2\rangle &= \sqrt{1/3}|-1\rangle_1|1/2\rangle_2 + \sqrt{2/3}|0\rangle_1|-1/2\rangle_2 \\ |3/2, -3/2\rangle &= |-1\rangle_1|-1/2\rangle_2 \\ |1/2, 1/2\rangle &= -\sqrt{1/3}|0\rangle_1|1/2\rangle_2 + \sqrt{2/3}|1\rangle_1|-1/2\rangle_2 \\ |1/2, -1/2\rangle &= -\sqrt{2/3}|-1\rangle_1|1/2\rangle_2 + \sqrt{1/3}|0\rangle_1|-1/2\rangle_2 \end{aligned}$$

- $m_e = 9.11 \times 10^{-31}$ kg
- $\hbar = 1.055 \times 10^{-34}$ J.s
- $\epsilon_0 = 8.854 \times 10^{-12}$ C^2/N.m^2
- $e = 1.6 \times 10^{-19}$ C