11/12

5.
$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} + \dot{z}\hat{k},$$

but $z = \pm ar$, so

$$\dot{\vec{r}} = \dot{r}\hat{k} + r\dot{\varphi}\hat{\varphi} \pm a\dot{r}\hat{k}$$

(a)
$$L = \frac{m}{2}(1+a^2)\dot{r}^2 + \frac{m}{2}r^2\dot{\varphi}^2 \mp mgar$$

(b)
$$p_r = m(1+a^2)\dot{r}, \quad p_\varphi = mr^2\dot{\varphi}$$

(c)
$$\frac{d}{dt}(mr^2\dot{\varphi}) = 0, \quad mr^2\dot{\varphi} = p_\varphi = \text{constant}$$

(d)
$$m(1+a^2)\ddot{r} = + mr\dot{\varphi}^2 \mp mga$$
$$= \frac{p_\varphi^2}{mr^3} \mp mga$$

(d) If there is a stable circular orbit, $r = r_0$, a constant, & then both $\dot{r} = \ddot{r} = 0$, so

$$0 = \frac{p_\varphi^2}{mr^3} \mp mga,$$

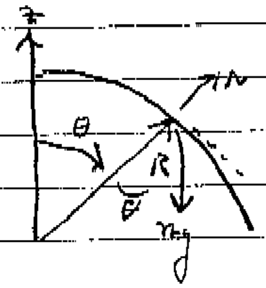
which is only possible for case ($\mp$),

$z = + ar$.

Classical Mechanics Jan '05
Solutions

1

1.(a)



Balance of forces along the normal to the surface:

$$mg\cos\theta - N = \frac{mv^2}{R}$$

Particle leaves surface when $N = 0$

Conservation of energy:

$$\frac{mv^2}{2} + mgR\cos\theta = mgR\cos\theta_0$$

solving for mv^2:

$$mv^2 = 2mgR(\cos\theta_0 - \cos\theta)$$

then $$N = mg(\cos\theta - 2(\cos\theta_0 - \cos\theta)) = \cancel{mg(\ldots)}$$

$$= mg(3\cos(\theta) - 2\cos(\theta_0))$$

particle leaves surface when

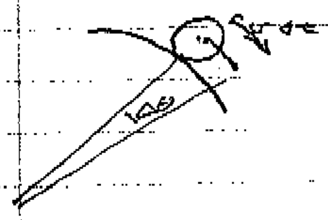
$$\cos(\theta) = \frac{2}{3}\cos(\theta_0)$$

2.1

1.(b)

$$mg\cos\theta - N = \frac{mv^2}{R+a}$$

$$\frac{mv^2}{2} + \frac{I\omega^2}{2} + mg(R+a)\cos\theta = mg(R+a)\cos\theta_0$$



$$\Delta\theta = \frac{v\Delta t}{R+a}$$

$$\Delta s = R\Delta\theta = a\omega\Delta t$$

$$\omega = \frac{R}{a\Delta t}\,\frac{v\Delta t}{R+a} = \frac{R}{R+a}\left(\frac{v}{a}\right)$$

$$\frac{m}{2}\left(1 + \frac{2}{5}\left(\frac{R}{R+a}\right)^2\right)v^2 = mg(R+a)(\cos\theta_0 - \cos\theta)$$

$$N = mg\cos\theta - \frac{2}{R+a}\,\frac{1}{1+\frac{2}{5}\left(\frac{R}{R+a}\right)^2}\,mg(R+a)(\cos\theta_0 - \cos\theta)$$

$= 0$ when

$$\cos\theta = \frac{\dfrac{2\cos\theta_0}{\left(1+\frac{2}{5}\left(\frac{R}{R+a}\right)^2\right)}}{1 + \dfrac{2}{1+\frac{2}{5}\left(\frac{R}{R+a}\right)^2}}$$

2.2

alternate 1.b

(a) $\vec{q}\,' = -\vec{q}$.

(b)
$$\left(\sqrt{m^2+p_0^2}+m\right)^2 - p_0^2 = (2M)^2$$
$$2m\sqrt{m^2+p_0^2} + 2m^2 = 4M^2$$
$$\sqrt{m^2+p_0^2} = \left(\frac{2M^2-m^2}{m}\right)$$
$$p_{min} = \left(\left[\frac{2M^2}{m} - m + m\right]\left[\frac{2M^2}{m} - m - m\right]\right)^{1/2}$$
$$= \frac{2M}{m}\sqrt{M^2-m^2}$$

(c)
$$\left(\sqrt{m^2+p_0^2}+m\right)^2 - p_0^2 = \left(2\sqrt{M^2+q^2}\right)^2$$
$$2\left(m^2 + m\sqrt{m^2+p_0^2}\right) = 4\left(M^2+q^2\right)$$
$$q = \left[\frac{m\sqrt{m^2+p_0^2} - 2M^2 + m^2}{2}\right]^{1/2}$$
$$= \left[\frac{mE_0 - 2M^2 + m^2}{2}\right]^{1/2}$$

(d) let $\beta = v_{cm}/c$, $E_0 = \sqrt{m^2+p_0^2}$

$$p_0' = \cancel{\gamma(E_0 - \beta p_0)} \quad \gamma(p_0 - \beta E_0)$$
$$-p_0' = \cancel{\gamma(E_0)} \quad \gamma(0 - \beta m)$$

adding, $p_0 = \beta(m+E_0)$, $\beta = \dfrac{p_0}{m+E_0}$

Let $E_q = \sqrt{M^2+q^2}$

2.3

$$P_x = \gamma(0 + \beta E_q) = P_x'$$

$$P_y = q = -P_y'$$

$$\tan\theta = \frac{q}{\gamma\beta E_q}$$

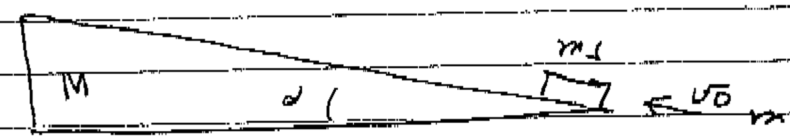
$$\gamma = \left(\sqrt{1 - \left(\frac{P_0}{m+E_0}\right)^2}\right)^{-1} = \frac{m+E_0}{2E_q}$$

$$\tan\theta = \frac{q}{\frac{(m+E_0)}{2}\frac{P_0}{m+E_0}} = \frac{2q}{P_0}$$

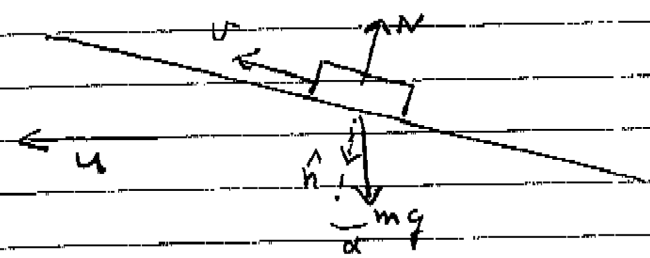
$$= \frac{2}{P_0}\sqrt{\frac{m\sqrt{m^2+p_0^2} - 2M^2 + m^2}{2}}$$

$$\theta' = -\theta$$

2.



initial impact: $(m_s+m)v_i = m v_0$, $v_i = \frac{m v_0}{m_s+m} = \frac{v_0}{3}$



let $\vec{u}$ = wedge's velocity.
$\ddot{u}$ is it's acceleration. It has a component opposite to N, of magnitude

$\dot{u}_n = \dot{u}\sin\alpha$

and, as the slider maintains contact with the wedge, this must also be the slider's acceleration in the $\hat{n}$ direction.

The slider's tangential velocity is $v_t = \vec{v}\cdot\hat{t}$

& $m_s \dot{v}_t = -m_s g \sin\alpha$

& it's normal acceleration must satisfy

$m_s \dot{u} \sin\alpha = m_s g \cos\alpha - N$

The Force N in the $\hat{n}$ direction acts on the wedge, but since the wedge rests on a table, only the component // to the table acts to accelerate the ~~slider,~~ wedge.

$$M\ddot{u} = N\sin\alpha$$

$$= (m_s g\cos\alpha - m_s\dot{u}\sin\alpha)\sin\alpha$$

we get

$$\dot{u} = \frac{m_s g\cos\alpha\sin\alpha}{M + m_s\sin^2\alpha}$$

The component of the wedge's velocity // to $\hat{n}$ is $u\sin\alpha$, and this must also be the $\hat{n}$ component of the slider's velocity, in order that the slider remain in contact.

∴ the slider's velocity vector is

$$\vec{v} = v\hat{s} + u\sin\alpha\,\hat{n},$$ where $\hat{s}$ is the unit vector // to the surface.

Conservation of energy is

$$\frac{m_s}{2}(v^2 + u^2\sin^2\alpha) + \frac{Mu^2}{2} + m_s g h = \frac{m_s v_1^2}{2}$$

$\vec{v}$ = velocity vector, relative to ground of ~~wedge~~ slider

$\vec{u}$ = " " of wedge "

$= -u\hat{x}$

$\vec{v} - \vec{u}$ = vel " of slider relative to wedge

$(\vec{v} - \vec{u}) \cdot \hat{n} = 0$. = condition that slider stay on surface.

~~At~~

At top of trajectory,

$\vec{v} - \vec{u} = 0$.

Then $\vec{v} \cdot \hat{n} = \vec{u} \cdot \hat{n} = u \sin\alpha$

& $\vec{v} \cdot \hat{t} = \vec{u} \cdot \hat{t} = u \cos\alpha \neq 0$.

(there is a point when $\vec{v} \cdot \hat{t} = 0$, but this must be after reaching the top. It occurs when the slider is on the way down)

5.1

at top, slider has instantaneous velocity $- u_f \hat{x}$

Energy,

$$\frac{m_s u_f^2}{2} + \frac{M u_f^2}{2} + m_s g h = \frac{m_s}{2} v_i^2$$

momentum:

$$(m_s + M) u_f = m_s v_i \cos\alpha$$

$$u_f = \frac{m_s v_i \cos\alpha}{m_s + M}$$

then

$$h = \frac{1}{m_s g}\left[\frac{m_s}{2} v_i^2 - \frac{1}{2}\frac{(m_s v_i \cos\alpha)^2}{m_s + M}\right]$$

$$= \frac{v_i^2}{2g}\left[1 - \frac{m_s \cos^2\alpha}{m_s + M}\right]$$

a

$$= \frac{v_i^2}{2g}\left[\frac{M + m_s \sin^2\alpha}{m_s + M}\right]$$

$$v_{tf} = u_f \cos\alpha$$

$$= v_i - g\sin\alpha\, t,$$

$$\text{so } t = \frac{1}{g\sin\alpha}\left[v_i - u_f\cos\alpha\right]$$

$$= \frac{v_i}{g\sin\alpha}\left[1 - \frac{m_s}{m_s+M}\cos^2\alpha\right]$$

(b) $$= \frac{v_i}{g\sin\alpha}\left(\frac{M + m_s\sin^2\alpha}{m_s+M}\right)$$

α, with

$$u_f = \dot{u}\, t$$

$$= \frac{m_s\, g\cos\alpha\sin\alpha}{M + m_s\sin^2\alpha}\;\frac{v_i}{g\sin\alpha}\;\frac{M+m_s\sin^2\alpha}{m_s+M}$$

(c) $$= \left(\frac{m_s}{m_s+M}\right)\frac{v_i}{g}\cos\alpha \, ,$$

in agreement with the above.

5-3

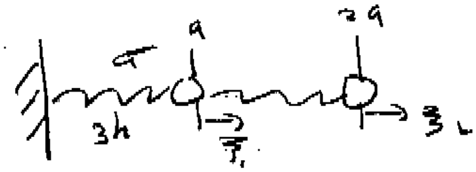
(d)

$$d_f = \ddot{x}\frac{t^2}{2}$$

$$= \frac{1}{2}\frac{m_s g \cos\alpha \sin\alpha}{M + m_s \sin^2\alpha}\left(\frac{v_i}{g\sin\alpha}\right)^2\left(\frac{M + m_s\sin^2\alpha}{M + m_s}\right)^2$$

$$= \frac{v_i^2}{2g}\frac{m_s}{(M+m_s)^2}(M + m_s\sin^2\alpha)\,\cancel{\tan\alpha}\,\cot\alpha$$

3.



(a) $$L = \frac{m\dot{\xi}_1^2}{2} + \frac{m\dot{\xi}_2^2}{2} - \frac{k}{2}\left(3\xi_1^2 + 2(\xi_2 - \xi_1)^2\right)$$

(b) Potential

$$V = \frac{k}{2}\,\xi^T A\,\xi,$$

with $$A = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

Eigenvalues

$$\lambda^2 - 7\lambda + 6 = 0,$$

$$\lambda = \frac{1}{2}\left(7 \pm \sqrt{49-24}\right) = \frac{1}{2}(7 \pm 5)$$

$$\lambda_+ = 6, \quad \lambda_- = 1.$$

Eigenvectors

$$u_+ = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad u_- = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Diagonalizing matrix, with $$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = S\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

7

$$L = \frac{m}{2}(\dot{\alpha}^2+\dot{\beta}^2) - \frac{k}{2}\left(6\alpha^2 + \beta^2\right)$$

Each mode has eigenfrequency,

$(+)$, $\omega_+ = \sqrt{\frac{6k}{m}}$, $(-)$, $\omega_- = \sqrt{\frac{k}{m}}$.

(c) $\xi_2(0) = b$, $\xi_1(0) = 0$.

$$\xi_1(t) = \frac{2}{\sqrt{3}}\alpha(t) + \frac{1}{\sqrt{3}}\beta(t)$$

$$\xi_2(t) = -\frac{1}{\sqrt{3}}\alpha(t) + \frac{1}{\sqrt{3}}\beta(t)$$

$$\begin{pmatrix}\xi_1\\ \xi_1\end{pmatrix}^{\cdot}(0) = 0, \quad \rightarrow \quad \dot{\alpha} = \dot{\beta} = 0.$$

$$\begin{pmatrix}0\\ b\end{pmatrix} = S\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix}$$

$$\begin{pmatrix}\alpha_0\\ \beta_0\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}0\\ b\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}-b\\ 2b\end{pmatrix}$$

$$\xi_1(t) = \frac{b}{3\sqrt{3}}\left(-2\cos(\omega_+ t) + 2\cos(\omega_- t)\right)$$

$$\xi_1(t) = \frac{b}{3}\left(+\cos(\omega_+ t) + 2\cos(\omega_- t)\right)$$

8

4. (a) $L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2 - \omega^2 r^2)$

$P_r = m\dot{r}, \quad P_\varphi = mr^2\dot{\theta}$

$H = \frac{1}{2m}P_r^2 + \frac{1}{2mr^2}P_\varphi^2 + \frac{m\omega^2}{2}r^2$

(b) $\frac{d}{dt}(mr^2\dot{\varphi}) = 0$

So $mr^2\dot{\varphi} = P_\varphi$ = constant (angular momentum)

$m\ddot{r} = mr\dot{\varphi}^2 - m\omega^2 r$

$\ddot{r} = \frac{P_\varphi^2}{m^2 r^3} - \omega^2 r$

(c) $E = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\varphi}^2}{2} + \frac{m\omega^2 r^2}{2}$

$= \frac{m\dot{r}^2}{2} + \frac{P_\varphi^2}{2mr^2} + \frac{m\omega^2}{2}r^2$

$\frac{dr}{dt} = \pm\left[\frac{2E}{m} - \frac{P_\varphi^2}{m^2r^2} - \omega^2 r^2\right]^{1/2}$

but $\frac{d\varphi}{dt} = \frac{P_\varphi}{mr^2}$,

dividing,

$$\frac{dr}{r^2 d\theta} \frac{1}{\left[\frac{2E}{m} - \frac{P_\phi^2}{m^2 r^2} - \omega^2 r^2\right]^{1/2}} = \frac{m}{P_\phi}$$

let $r = \frac{1}{u}$,

$$\frac{du}{\sqrt{\frac{2E}{m} - \frac{P_\phi^2}{m^2} u^2 - \frac{\omega^2}{u^2}}} = + \frac{m}{P_\phi} d\phi$$

let $\frac{2E}{m} = 2\alpha$, $\frac{P_\phi^2}{m^2} = \beta$, $\omega^2 = \gamma$,

$u = \sqrt{s}$.

$$\frac{ds}{2\sqrt{2\alpha s - \beta s^2 - \gamma}} = \frac{m}{P_\theta} d\phi$$

$$\frac{ds}{\left[\frac{\alpha^2}{\beta} - \gamma - \beta\left(s - \frac{\alpha}{\beta}\right)^2\right]^{1/2}} = \frac{2m}{P_\phi} d\phi$$

$$s = \frac{\alpha}{\beta} + \sqrt{\frac{\frac{\alpha^2}{\beta} - \gamma}{\beta}}\, \zeta$$

$$\frac{d\zeta}{\left[1 - \zeta^2\right]^{1/2}} = \frac{2m}{P_\theta} \sqrt{\beta}\, d\phi$$
$$= 2\, d\phi$$

$\zeta = \sin x$

$dx = 2\, d\phi$, $x = 2\phi$.

so $$\frac{1}{r} = \sqrt{\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2/\beta - 1}{\beta}}\;\sin 2\varphi}$$

$$r = \frac{\sqrt{\beta}}{\left[\alpha + \sqrt{\alpha^2 - \beta}\;\sin 2\varphi\right]^{1/2}}$$

$$= \frac{p_\varphi / m}{\left[\frac{E}{m} + \sqrt{\left(\frac{E}{m}\right)^2 - \left(\frac{p_\varphi}{m}\right)^2 \omega^2}\;\sin 2\varphi\right]^{1/2}}$$

$$\cancel{= \frac{p_\varphi}{\left[E + \sqrt{E^2 - (p_\varphi \omega)^2}\;\sin 2\varphi\right]}}$$

$$= \frac{p_\varphi / \sqrt{m}}{\sqrt{E + \sqrt{E^2 - (p_\varphi \omega)^2}\;\sin(2\varphi)}}$$

~~13~~ 12

(e) $r_0 = \left[\left(\frac{p_\varphi}{m}\right)^2 \frac{1}{g_a} \right]^{1/3}$,

but $r_0^2 \omega = \frac{p_\varphi}{m}$, so

$r_0 = r_0^{4/3} \left[\frac{\omega^2}{g_a} \right]^{1/3}$,

+

$r_0 = \frac{g_a}{\omega^2}$.

(f) Let $r = r_0 + \delta r$, $\delta r, \delta \dot{r}, \delta \ddot{r}$ small of first order.

$m(1+a^2)\,\delta \ddot{r} = -\frac{3\, p_\varphi^2}{m\, r_0^4}\,\delta r$,

and $\delta r = A \sin(\omega_{osc} t)$,

$\omega_{osc} = \sqrt{3\left(\frac{p_\varphi}{m}\right)^2 \frac{1}{(1+a^2)\, r_0^4}}$

$= \left[3\, r_0^4\, \omega^2 \frac{1}{(1+a^2)\, r_0^4} \right]^{1/2}$

$= \omega \sqrt{\frac{3}{1+a^2}}$